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On the magnetohydrodynamic limits of the ideal two-fluid plasma equations

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The two-fluid plasma equations describing a magnetized plasma, originating from truncating moments of the Vlasov-Boltzmann equation, are increasingly used to describe an ion-electron plasma whose transport phenomena occur on a time scale slower and a length scale longer than those of particle collisions. A similar treatment under more stringent constraints gives the single-fluid magnetohydrodynamic (MHD) equations for low-frequency macroscopic processes. Since both stem from kinetic theory, the two-fluid plasma and MHD equations are necessarily related to each other. Such a connection is often established via *ad hoc* physical reasoning without a firm analytical foundation. Here, we perform a sequence of formal expansions for the dimensionless ideal two-fluid plasma equations with respect to limiting values of the speed-of-light c, the ion-to-electron mass ratio M, and the plasma skin depth d_S. Several different closed systems of equations result, including separate systems for each limit applied in isolation and those resulting from limits applied in combination, which correspond to the well-known Hall-MHD and single-fluid ideal MHD equations. In particular, it is shown that while the zeroth-order description corresponding to the $c \to \infty$ limit, with M and d_S fixed, is strictly charge neutral, it nonetheless uniquely determines the perturbation charge non-neutrality at the first order. Furthermore, the additional $M \to \infty$ limit is found to be not required to obtain the single-fluid MHD equations despite being essential for the Hall-MHD system. The hierarchy of systems presented demonstrates how plasmas can be appropriately modeled in situations where only one of the limits applies, which lie in the parameter space in between where the two-fluid plasma and Hall-MHD models are appropriate. *Published by AIP Publishing.* https://doi.org/10.1063/1.5067387

I. INTRODUCTION

Starting with the Vlasov-Boltzmann equation in classic kinetic theory where a Maxwellian velocity distribution function is assumed, the two-fluid equations for a plasma emerge from truncating the moment series.[1–3] Closure for such a five moment system is obtained, provided that the hydrodynamic time scale of interest, τ_H, is much slower than the thermal relaxation time scales, $\tau_{e,i}$, for both electrons and ions in a two-fluid plasma (2FP), that is, $\tau_{e,i} \ll \tau_H$, for a fluidic description for the plasma to be applicable. For a plasma with singly charged ion and equal species temperatures, explicit estimates for these relaxation times are found as[4]

$$\tau_e = 6\pi\sqrt{2\pi}\epsilon_0^2 \frac{m_e^{1/2}(k_B T)^{3/2}}{\ln \Lambda e^4 n_e}, \quad \tau_i = 6\pi\sqrt{2\pi}\epsilon_0^2 \frac{m_i^{1/2}(k_B T)^{3/2}}{\ln \Lambda e^4 n_i}, \tag{1}$$

where $m_{e,i}$ are the electron and ion masses, e is the electron charge, n_i is the ion number density, T is the temperature, k_B is the Boltzmann constant, ϵ_0 is the vacuum permittivity, and $\ln \Lambda$ is the Coulomb potential which evaluates to be of order $O(10)$ in most of the plasmas. The two-fluid plasma model, henceforth denoted as 2FP, is particularly relevant when the characteristic length scale is comparable to the ion skin depth, and the characteristic time scale is comparable to the ion cyclotron period.[5]

Under mild restrictions, a wide range of plasma applications can be suitably described by the ideal 2FP equations, where dissipative effects are neglected. The validity of this simplification requires expressions for the transport coefficients derived from the Chapman-Enskog expansion[3] where distribution functions deviated from the local thermal equilibrium are expanded in powers of small parameters $\epsilon_{e,i} = \tau_{e,i}/\tau_H \ll 1$. Goedbloed and Poedts[4] summarized that the viscosity and thermal conductivity can be neglected if the dissipative diffusion occurs at time scales sufficiently large compared to τ_H, which is generally satisfied over macroscopic geometries. The ideal 2FP model is valid for τ_H smaller than the diffusion or dissipation time scale τ_D, i.e., $\tau_H \ll \tau_D$. Ion diffusion processes appear in two versions: one is parallel to the magnetic field lines $\tau_{D,||,i} \propto (v_{th,i}^2 \tau_i)^{-1}$, where $v_{th,i}$ is the ion thermal speed, and the other that is perpendicular to the field lines is $\tau_{D,\perp,i} \propto \tau_{D,||,i}(\Omega_i \tau_i)^2$, where Ω_i is the ion cyclotron frequency. For most systems of interest, both parallel and perpendicular diffusion time scales are generally much larger than τ_H.[4]

The resistivity, due to ion-electron momentum transfer, gives a time scale, τ_R, which can be estimated as

$$\tau_R = \left(\frac{a}{\delta_e}\right)^2 \tau_e, \tag{2}$$

where a is a scale associated with the plasma system size and δ_e is the electron skin depth. Generally, the resistivity is negligible if $\tau_H \ll \tau_R$.[4]

Finally, heat flux due to interspecies collisions is considered small so long as τ_H differs significantly from the overall, longest thermal equilibration time, τ_{eq}, which scales as $\tau_{eq} \sim \sqrt{M}\tau_i \sim M\tau_e$, with $M \equiv m_i/m_e \gg 1$, assuming that Eq. (1) holds. It is also noted that the large M assumption is essential in obtaining these time scale estimates from the Landau collision integral. Thus, the ideal 2FP model considers each species to be in its own kinetic-collisional equilibrium but not necessarily with the same temperatures, with the ion and electron temperatures equilibrating over the time scale τ_{eq}. Later we will show that the large $M \gg 1$ assumption is not essential in obtaining the single fluid magnetohydrodynamic (MHD) equations. Considering the fact we stated above that the ideal 2FP model is derived under $M \gg 1$, we may have an apparent contradiction. However, this is resolved because the heat flux due to interspecies collision can also be neglected for the hydrodynamic time scale exceeding τ_{eq}, and here, we have an equilibration of ion and electron temperatures and the ideal 2FP system of equations is still valid. Hence, for such cases of interest, $\tau_H \gg \tau_{eq}$, the requirement of $M \gg 1$ may be relaxed.

Although less general, reduced plasma models such as the Hall-MHD and ideal MHD equations are more popular than the 2FP description for modeling low-frequency processes owing to their relative simplicity. When approached from the kinetic theory or the 2FP equations, the foundation basis underlying these reduced systems often relies on tailored physical approximations. For example, it is commonly believed that in order to obtain the MHD equations, one needs a series of independent assumptions including large speed-of-light, charge neutrality, a large ion-to-electron mass ratio, and a small Larmor radius.[6] Similarly, the Hall-MHD model is obtained by relaxing specific constraints on the generalized Ohm's law posed in MHD.[7] An analytically consistent treatment bridging the two-fluid system with the various MHD formulations appears to have received little attention.

This paper is intended to provide a mathematically firm derivation of various limiting forms of the ideal 2FP equations, including both the Hall-MHD and MHD equations, by taking a sequence of formal asymptotic limits with respect to suitably defined dimensionless parameters, namely, large speed-of-light c, large ion-electron mass ratio M, small plasma skin depth d_S, and finite plasma parameter β. The associated homogeneous dispersion relation for each derived limiting system is also calculated analytically and compared with the existing results where applicable.[4,5,8,9] Asymptotic analysis for extreme values of the frequency and wave number is performed in order to provide physical insights into the appropriate wave-propagation physics. Conveniently, none of the aforementioned conditions for the ideal fluid assumption are violated in the limiting processes because a large relative speed-of-light would drive a slow hydrodynamic process to ensure $\epsilon_{e,i} \ll 1$. Furthermore, it can be shown from Eq. (2) that if d_S is defined using mass m_i, then $\tau_R \sim M/d_S^2 \tau_e$ so that both the large mass ratio and small skin depth would imply a long resistance delay time to guarantee $\tau_H \ll \tau_R$.

This paper is structured as follows: Section II introduces the non-dimensional, ideal two-fluid plasma equations written in the center-of-mass frame and studies its dispersion relation. Sections III, IV, and V individually examine the infinite c, large M, and the small plasma skin depth d_S limits of the two-fluid system, leading to three corresponding closed sets of equations in the limit. In Sec. VI, two of the three limits are applied consecutively to obtain the well-known Hall-MHD and ideal MHD equations. Dispersion relations derived for all limiting forms of the two-fluid equations are analytically determined and asymptotically compared.

II. NON-DIMENSIONAL IDEAL TWO-FLUID PLASMA EQUATIONS

A. Equations of motion

We begin with the two-fluid equations of an ideal plasma (2FP) given by[10]

$$
\begin{aligned}
\frac{\partial \rho_\alpha}{\partial t} + \nabla\cdot(\rho_\alpha \mathbf{u}_\alpha) &= 0,\\
\frac{\partial \rho_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla\cdot(\rho_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + p_\alpha \mathbf{I}) &= n_\alpha q_\alpha(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}),\\
\frac{\partial \varepsilon_\alpha}{\partial t} + \nabla\cdot((\varepsilon_\alpha + p_\alpha)\mathbf{u}_\alpha) &= n_\alpha q_\alpha \mathbf{E}\cdot\mathbf{u}_\alpha,\\
\frac{\partial \mathbf{B}}{\partial t} + \nabla\times\mathbf{E} &= \mathbf{0},\\
\frac{\partial \mathbf{E}}{\partial t} - c^2\nabla\times\mathbf{B} &= -\frac{1}{\epsilon_0}\sum_\alpha n_\alpha q_\alpha \mathbf{u}_\alpha,\\
\nabla\cdot\mathbf{E} &= \frac{1}{\epsilon_0}\sum_\alpha n_\alpha q_\alpha,\\
\nabla\cdot\mathbf{B} &= 0,
\end{aligned}
\tag{3}
$$

where

$$
\rho_\alpha = n_\alpha m_\alpha, \quad p_\alpha = n_\alpha k_B T_\alpha, \quad \varepsilon_\alpha = \frac{p_\alpha}{\gamma-1} + \frac{\rho_\alpha|\mathbf{u}_\alpha|^2}{2}. \tag{4}
$$

Here, $\mathbf{x}$ is the position vector and t is the time. The subscript α denotes ion or electron species, with ρ_α being the mass density, m_α the particle mass, n_α the species number density, $\mathbf{u}_\alpha$ the species velocity, q_α the particle charge, T_α the species temperature, p_α the pressure, and ε the thermal energy. Separate equations-of-state, with k_B being the Boltzmann constant and γ the specific heat ratio, are applied for both ions and electrons, meaning that each species is in its own kinetic-collisional equilibrium but not necessarily with the same temperatures. In Maxwell's equations, $\mathbf{B}$ and $\mathbf{E}$ are the magnetic and electric fields, respectively, and the speed-of-light is given by $c = (\mu_0\epsilon_0)^{-1/2}$ with μ_0 being the permeability of free space and ϵ_0 the vacuum permittivity.

In order to obtain self-consistent limits, we introduce a non-dimensionalization scheme where reference scales are chosen for length $L_{\rm ref}$, mass $m_{\rm ref}$, number density $n_{\rm ref}$, velocity $u_{\rm ref}$, charge $q_{\rm ref}$, and magnetic field $B_{\rm ref}$. Dimensionless variables are defined accordingly

$$\hat{\mathbf{x}} = \frac{\mathbf{x}}{L_{\text{ref}}}, \quad \hat{t} = \frac{t}{L_{\text{ref}}/u_{\text{ref}}}, \quad \hat{\rho}_\alpha = \frac{\rho_\alpha}{n_{\text{ref}} m_{\text{ref}}}, \quad \hat{m}_\alpha = \frac{m_\alpha}{m_{\text{ref}}},$$
$$\hat{n}_\alpha = \frac{n_\alpha}{n_{\text{ref}}}, \quad \hat{\mathbf{u}}_\alpha = \frac{\hat{\mathbf{u}}}{u_{\text{ref}}}, \quad \hat{q}_\alpha = \frac{q_\alpha}{q_{\text{ref}}}, \quad \hat{p}_\alpha = \frac{p_\alpha}{n_{\text{ref}} m_{\text{ref}} u_{\text{ref}}^2},$$
$$\hat{\varepsilon}_\alpha = \frac{\varepsilon_\alpha}{n_{\text{ref}} m_{\text{ref}} u_{\text{ref}}^2}, \quad \hat{\mathbf{B}} = \frac{\mathbf{B}}{B_{\text{ref}}}, \quad \hat{\mathbf{E}} = \frac{\mathbf{E}}{u_{\text{ref}} B_{\text{ref}}}. \tag{5}$$

It is noted, that assuming that plasma is initially magnetized, an independent scale for magnetic field B_{ref} is introduced. Additionally, instead of defaulting the speed-of-light, c, as the reference velocity, an independent characteristic speed, u_{ref}, is allowed to scale c and give $\hat{c}$.

Therefore, expressed in terms of the dimensionless variables with the hat symbol dropped henceforth for brevity, the non-dimensional ideal two-fluid equations are given by

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \mathbf{u}_\alpha) = 0, \tag{6}$$

$$\frac{\partial \rho_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + p_\alpha \mathbf{I}) = \frac{n_\alpha q_\alpha}{d_L} (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}), \tag{7}$$

$$\frac{\partial \varepsilon_\alpha}{\partial t} + \nabla \cdot ((\varepsilon_\alpha + p_\alpha) \mathbf{u}_\alpha) = \frac{n_\alpha q_\alpha}{d_L} \mathbf{E} \cdot \mathbf{u}_\alpha, \tag{8}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0}, \tag{9}$$

$$\frac{\partial \mathbf{E}}{\partial t} - c^2 \nabla \times \mathbf{B} = -\frac{d_L}{d_D^2} \sum_\alpha n_\alpha q_\alpha \mathbf{u}_\alpha, \tag{10}$$

$$\nabla \cdot \mathbf{E} = \frac{d_L}{d_D^2} \sum_\alpha n_\alpha q_\alpha, \tag{11}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{12}$$

where

$$\rho_\alpha = n_\alpha m_\alpha, \quad \varepsilon_\alpha = \frac{p_\alpha}{\gamma - 1} + \frac{\rho_\alpha |\mathbf{u}_\alpha|^2}{2}, \tag{13}$$

and

$$d_D \equiv \sqrt{\frac{u_{\text{ref}}^2 \epsilon_0 m_{\text{ref}}}{n_{\text{ref}} q_{\text{ref}}^2 L_{\text{ref}}^2}} = \frac{1}{q_{\text{ref}} c L_{\text{ref}}} \sqrt{\frac{m_{\text{ref}}}{n_{\text{ref}} \mu_0}}, \quad d_L \equiv \frac{u_{\text{ref}} m_{\text{ref}}}{q_{\text{ref}} B_{\text{ref}} L_{\text{ref}}}, \tag{14}$$

are the dimensionless Debye length and Larmor radius, respectively.

Since the Debye length d_D varies with c, it is more convenient to introduce the plasma skin depth, d_S,

$$d_S \equiv \frac{1}{q_{\text{ref}} L_{\text{ref}}} \sqrt{\frac{m_{\text{ref}}}{\mu_0 n_{\text{ref}}}}, \tag{15}$$

which measures the distance of which electromagnetic (EM) waves can penetrate and the plasma parameter β

$$\beta \equiv \frac{2\mu_0 n_{\text{ref}} m_{\text{ref}} u_{\text{ref}}^2}{B_{\text{ref}}^2}, \tag{16}$$

which measures the relative size of thermal energy over magnetic energy. Using the following mapping between the two parameter sets

$$d_D = \frac{d_S}{c}, \quad d_L = \sqrt{\frac{\beta}{2}} d_S, \tag{17}$$

the behavior of the ideal two-fluid system can be fully characterized by the four independent non-dimensional parameters, namely, c, M, d_S, and β.

B. Center-of-mass representation

We proceed with the singly charged ion case where $q_i = e$ is the dimensionless proton charge. For the purpose of enabling a clear and physically insightful asymptotic analysis, it is convenient to transform the primitive variables for each species, $(\rho_{i,e}, p_{i,e}, \mathbf{u}_{i,e})^T$, into their corresponding counterparts viewed from the center-of-mass frame, by defining the total mass density ρ, charge density ρ_c, net pressure p, center-of-mass velocity $\mathbf{u}$, and current $\mathbf{j}$. The change in variables is then

$$\begin{aligned} &\rho = \rho_i + \rho_e, \quad \rho_c = e(n_i - n_e), \quad p = p_i + p_e, \\ &\mathbf{u} = \frac{\rho_i \mathbf{u}_i + \rho_e \mathbf{u}_e}{\rho_i + \rho_e}, \quad \mathbf{j} = e(n_i \mathbf{u}_i - n_e \mathbf{u}_e). \end{aligned} \tag{18}$$

The original species variables can be recovered by

$$\begin{aligned} \rho_i &= \frac{M\rho + \rho_c m_i/e}{1+M}, \quad \rho_e = \frac{\rho - \rho_c m_i/e}{1+M}, \\ \mathbf{u}_i &= \frac{M\rho \mathbf{u} + \mathbf{j} m_i/e}{M\rho + \rho_c m_i/e}, \quad \mathbf{u}_e = \frac{\rho \mathbf{u} - \mathbf{j} m_i/e}{\rho - \rho_c m_i/e}, \end{aligned} \tag{19}$$

where

$$M = \frac{m_i}{m_e} \tag{20}$$

is the particle mass ratio. It might be tempting at this point to set m_{ref} to be the ion mass and q_{ref} to be the proton charge, giving $m_i = e = 1$. In general, a specific choice for reference scales is not necessarily made until a specific flow is considered so that we here reserve the capability to easily convert the dimensionless equations back to their dimensional form.

Substituting (19) into the 2FP equations, one obtains the following conservation laws:

- mass and charge density continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{21}$$

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot \mathbf{j} = 0, \tag{22}$$

- momentum and current conservation

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + p\mathbf{I}) + \nabla \cdot \mathbf{f}_{mom} = \frac{\sqrt{2}}{\sqrt{\beta} d_S} (\mathbf{j} \times \mathbf{B} + \rho_c \mathbf{E}), \tag{23}$$

$$\begin{aligned} &\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot \left(\mathbf{u}\mathbf{j} + \mathbf{j}\mathbf{u} - \frac{m_i}{e\rho} \mathbf{j}\mathbf{j} \right) + \nabla \cdot \mathbf{f}_{cur} \\ &= \frac{\sqrt{2} e^2 M \rho (\mathbf{E} + \mathbf{u} \times \mathbf{B})}{\sqrt{\beta} d_S m_i^2} + \mathbf{s}_{cur}, \end{aligned} \tag{24}$$

where

$$
\begin{aligned}
\mathbf{f}_{mom} &= \rho\frac{m_i^2}{e^2}\frac{\rho_c^2\mathbf{uu}-\rho_c\mathbf{uj}-\rho_c\mathbf{ju}+\mathbf{jj}}{(M\rho+\rho_c m_i/e)(\rho-\rho_c m_i/e)},\\
\mathbf{f}_{cur} &= \frac{m_i\left[\mathbf{jj}\left(e^2\rho^2-\rho_c m_i(\rho_c m_i+eM\rho)\right)+e\rho\rho_c(\mathbf{ju}+\mathbf{uj})(\rho_c m_i+e(M-1)\rho)\right]}{e\rho(e\rho-\rho_c m_i)(\rho_c m_i+eM\rho)}\\
&\quad-\frac{e^3M\rho^3\rho_c\mathbf{uu}}{e\rho(e\rho-\rho_c m_i)(\rho_c m_i+eM\rho)}+\frac{e(p-(M+1)p_e)}{m_i}\mathbf{I},\\
\mathbf{s}_{cur} &= \frac{\sqrt{2}(1-M)e}{\sqrt{\beta}d_S m_i}(\rho_c\mathbf{E}+\mathbf{j}\times\mathbf{B}),
\end{aligned}
\tag{25}
$$

- total energy conservation

$$
\frac{\partial\mathcal{E}}{\partial t}+\frac{\partial\mathcal{E}_{erg}}{\partial t}+\nabla\cdot\left((\mathcal{E}_h+p)\mathbf{u}+\frac{2}{\beta}\mathbf{E}\times\mathbf{B}\right)+\nabla\cdot\mathbf{f}_{erg}=0,
\tag{26}
$$

where

$$
\begin{aligned}
&\mathcal{E}=\mathcal{E}_h+\frac{|\mathbf{B}|^2}{\beta},\quad \mathcal{E}_h=\frac{p}{\gamma-1}+\frac{1}{2}\rho|\mathbf{u}|^2,\quad \mathcal{E}_{erg}=\rho\frac{m_i^2}{e^2}\frac{(|\mathbf{u}|\rho_c-|\mathbf{j}|)^2}{2(1+M)^2\rho_i\rho_e}+\frac{|\mathbf{E}|^2}{\beta c^2},\\
&\mathbf{f}_{erg}=\frac{\rho m_i^2(|\mathbf{j}|-|\mathbf{u}|\rho_c)\left[\rho_c\left(\rho_c m_i\left(m_i(|\mathbf{u}|\rho_c+|\mathbf{j}|)+2e(M-1)\rho|\mathbf{u}|\right)-3e^2M\rho^2|\mathbf{u}|\right)+e^2|\mathbf{j}|M\rho^2\right]}{2(e\rho-\rho_c m_i)^2(\rho_c m_i+eM\rho)^2}\mathbf{u}\\
&\qquad-\frac{\rho m_i^2(|\mathbf{j}|-|\mathbf{u}|\rho_c)\left(e(M-1)\rho m_i(|\mathbf{u}|\rho_c+|\mathbf{j}|)+2|\mathbf{j}|\rho_c m_i^2-2e^2M\rho^2|\mathbf{u}|\right)}{2(e\rho-\rho_c m_i)^2(\rho_c m_i+eM\rho)^2}\mathbf{j}\\
&\qquad-\frac{\gamma}{\gamma-1}\frac{m_i(\mathbf{j}-\mathbf{u}\rho_c)(p\rho_c m_i+e(M+1)\rho p_e-ep\rho)}{(e\rho-\rho_c m_i)(\rho_c m_i+eM\rho)},
\end{aligned}
\tag{27}
$$

- electron pressure equation

$$
\frac{\partial p_e}{\partial t}+\left(\mathbf{u}-\frac{m_i}{e\rho}\mathbf{j}\right)\cdot\nabla p_e+\gamma p_e\nabla\cdot\left(\mathbf{u}-\frac{m_i}{e\rho}\mathbf{j}\right)+S_{p_e}=0,
\tag{28}
$$

where

$$
S_{p_e}=\frac{\rho_c m_i}{e\rho-\rho_c m_i}\left[\left(\mathbf{u}-\frac{m_i}{e\rho}\mathbf{j}\right)\cdot\nabla p_e+\gamma p_e\nabla\cdot\left(\mathbf{u}-\frac{m_i}{e\rho}\mathbf{j}\right)\right],
\tag{29}
$$

and the Maxwell equations

$$
\frac{\partial\mathbf{B}}{\partial t}+\nabla\times\mathbf{E}=\mathbf{0},
\tag{30}
$$

$$
\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}-\nabla\times\mathbf{B}=-\frac{\sqrt{\beta}}{\sqrt{2}d_S}\mathbf{j},
\tag{31}
$$

$$
\frac{1}{c^2}\nabla\cdot\mathbf{E}=\frac{\sqrt{\beta}}{\sqrt{2}d_S}\rho_c,
\tag{32}
$$

$$
\nabla\cdot\mathbf{B}=0.
\tag{33}
$$

Hence, Eqs. (21)–(33) give the center-of-mass representation of the ideal 2FP equations. Symmetry breaks down in this form when the equations of motion are written for macroscopic quantities, for instance, the current, resulting in algebraically formidable expressions. Nonetheless, such preparation is necessary to enable a discussion of various limits of the 2FP system with respect to c, M, d_S, and β, as well as the physical implications of these limits.

Among the various 2FP contractions discussed in the sequel, it is useful to define a distinction between what we refer to as a "plasma" (P) model, which supports nonvanishing charge separation and a wide spectrum of waves including electromagnetic waves, and a "magnetohydrodynamic" (MHD) model which we will define, as is conventional, to satisfy charge quasi-neutrality.[6]

C. Dispersion relation for 2FP

1. Linearization around a homogeneous equilibrium

It is insightful to analyze the waves permitted by the 2FP system, as an example of the general procedure that will be repeatedly used. We consider perturbation away from a homogeneous stationary background equilibrium (subscripted by zero), where $\tilde{\mathbf{u}}_0=\tilde{\mathbf{j}}_0=\tilde{\mathbf{E}}_0=\mathbf{0}$, $\tilde{\rho}_{c0}=0$, and $\tilde{\mathbf{B}}_0$, $\tilde{p}_0$, $\tilde{p}_{e0}$, $\tilde{\rho}_0$ define the unperturbed constant state. Here, the tilde symbol refers variables to their dimensional form. This leads to the following natural choice for reference scales:

$$
\begin{aligned}
&B_{\mathrm{ref}}=|\tilde{\mathbf{B}}_0|,\quad u_{\mathrm{ref}}=\sqrt{\frac{\gamma\tilde{p}_0}{n_{\mathrm{ref}}m_{\mathrm{ref}}}},\quad m_{\mathrm{ref}}=\tilde{m}_i+\tilde{m}_e,\\
&n_{\mathrm{ref}}=\tilde{n}_{i0}=\tilde{n}_{e0},\quad q_{\mathrm{ref}}=\tilde{e}.
\end{aligned}
\tag{34}
$$

It immediately follows that $m_i = M/(M+1)$ and $e = 1$. Since the background equilibrium is stationary, a velocity scale is conveniently found through the initial pressure in the form of speed-of-sound. The benefit of this choice is that the background pressure can be normalized in the linearized equations. Additionally, the length scale, L_{ref}, in this case, has to be inferred from a known value of d_S, according to Eq. (15).

The non-dimensional field variables can now be expanded as a regular perturbation around the background solution, giving

$$\begin{aligned} &\rho = 1+\rho', \quad \rho_c = \rho_c', \quad \mathbf{u} = \mathbf{u}', \quad \mathbf{j} = \mathbf{j}', \\ &p = \frac{1}{\gamma} + p', \quad p_e = \frac{\alpha}{\gamma} + p_e', \quad \mathbf{B} = \mathbf{b} + \mathbf{B}', \quad \mathbf{E} = \mathbf{E}', \end{aligned} \tag{35}$$

where $\alpha \equiv \tilde{p}_{e0}/\tilde{p}_0 \in (0,1)$ is the initial electron temperature fraction and $\mathbf{b}$ is the unit vector in the direction of the background magnetic field, $\tilde{\mathbf{B}}_0$. Substituting (35) into (21)–(33), using the Faraday law to expose pressure from the energy equation, and retaining terms linear in perturbation quantities give the linearized 2FP equations

$$\begin{aligned} &\frac{\partial \rho'}{\partial t} + \nabla\cdot\mathbf{u}' = 0, \quad \frac{\partial \rho_c'}{\partial t} + \nabla\cdot\mathbf{j}' = 0, \\ &\frac{\partial \mathbf{u}'}{\partial t} + \nabla p' = \frac{1}{d_S}\sqrt{\frac{2}{\beta}}\mathbf{j}'\times\mathbf{b}, \\ &\frac{\partial \mathbf{j}'}{\partial t} + \frac{1+M}{M}\nabla p' - \frac{(1+M)^2}{M}\nabla p_e' \\ &\quad = \frac{1}{Md_S}\sqrt{\frac{2}{\beta}}\Big[(1+M)^2(\mathbf{E}'+\mathbf{u}'\times\mathbf{b}) + (1-M^2)\mathbf{j}'\times\mathbf{b}\Big], \\ &\frac{\partial p'}{\partial t} + \nabla\cdot\mathbf{u}' - \frac{\alpha(1+M)-1}{1+M}\nabla\cdot\mathbf{j}' = 0, \\ &\frac{\partial p_e'}{\partial t} + \alpha\nabla\cdot\mathbf{u}' - \frac{\alpha M}{1+M}\nabla\cdot\mathbf{j}' = 0, \\ &\frac{\partial \mathbf{B}'}{\partial t} + \nabla\times\mathbf{E}' = 0, \quad \frac{1}{c^2}\frac{\partial \mathbf{E}'}{\partial t} - \nabla\times\mathbf{B}' = -\frac{1}{d_S}\sqrt{\frac{\beta}{2}}\mathbf{j}'. \end{aligned} \tag{36}$$

Both the divergence constrains on $\mathbf{E}$ and $\mathbf{B}$ have been omitted with the consequence that spurious stationary waves could be introduced together with the genuine entropy waves that are also stationary.[4] However, neither of these null solutions are of interest, and so, only positive wave frequencies are considered throughout the analysis.

2. Plane wave solutions

We seek plane wave solutions of the form

$$\xi(\mathbf{x},t) = |\xi|\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t), \tag{37}$$

where ξ represents a general function of space and time. Both the wave-number vector $\mathbf{k}$ and the angular frequency ω are dimensionless, scaled by $1/L_{\text{ref}}$ and $u_{\text{ref}}/L_{\text{ref}}$, respectively. The Cartesian coordinates are oriented such that $\mathbf{b}$ is along the z-direction, and $\mathbf{k}$ lies in the x, z-plane

$$\mathbf{b} = (0,\,0,\,1), \quad \mathbf{k} = (k_\perp,\,0,\,k_\parallel), \quad \lambda \equiv \frac{k_\parallel}{|\mathbf{k}|} = \frac{k_\parallel}{k}, \tag{38}$$

where $k_\perp$ and $k_\parallel$ are the wave number components perpendicular and parallel to the background magnetic field, respectively, and λ gives the cosine of the angle between $\mathbf{b}$ and $\mathbf{k}$.

Since ρ_1 and ρ_{c1} decouple from the system, they can be consistently dropped together with the two divergence constraints on $\mathbf{E}'$ and $\mathbf{B}'$. Using the ansatz Eq. (37), the remaining equations lead to an algebraic system for p', p_e', $\mathbf{u}'$, $\mathbf{j}'$, $\mathbf{B}'$, and $\mathbf{E}'$. Here, we note that there are two wave families that are considered "marginal," i.e., corresponding to $\omega = 0$.[4] These waves correspond to spatial distributions of ion and electron densities and pressure balanced by the longitudinal electric field. By excluding marginal waves for which $\omega = 0$, after some algebra, all the other unknowns can be expressed in terms of $\mathbf{u}'$ and $\mathbf{E}'$ only, giving a reduced eigenvalue problem,

$$\begin{bmatrix} i(k_\perp^2-\omega^2) & 0 & ik_\perp k_\parallel & a_1l_1k_\perp^2\omega & \dfrac{2i(c^2k^2-\omega^2)}{c^2\beta} & a_1l_1k_\perp k_\parallel\omega \\ 0 & -i\omega^2 & 0 & \dfrac{2i(\omega^2-c^2k_\parallel^2)}{c^2\beta} & 0 & \dfrac{2ik_\perp k_\parallel}{\beta} \\ ik_\perp k_\parallel & 0 & i(k_\parallel^2-\omega^2) & a_1l_1k_\perp k_\parallel\omega & 0 & a_1l_1k_\parallel^2\omega \\ -ia_2k_\perp^2 & -l_2\omega & -ia_2k_\perp k_\parallel & l_1\omega(\omega^2-a_3k_\perp^2-c^2k_\parallel^2)-l_2\omega & il_3(\omega^2-c^2k^2) & a_4l_1k_\perp k_\parallel\omega \\ l_2\omega & 0 & 0 & il_3(c^2k_\parallel^2-\omega^2) & l_1\omega(\omega^2-c^2k^2)-l_2\omega & -il_3c^2k_\perp k_\parallel \\ -ia_2k_\perp k_\parallel & 0 & -ia_2k_\parallel^2 & a_4l_1k_\perp k_\parallel\omega & 0 & l_1\omega(\omega^2-a_3k_\parallel^2-c^2k_\perp^2)-l_2\omega \end{bmatrix} \begin{bmatrix} \mathbf{u}' \\ \\ \\ \mathbf{E}' \\ \\ \\ \end{bmatrix} = 0, \tag{39}$$

where

$$a_1 = \frac{\alpha + \alpha M - 1}{M+1}, \quad a_2 = \frac{(M+1)(\alpha + \alpha M - 1)}{M},$$
$$a_3 = \frac{-\alpha + \alpha M^2 + 1}{M}, \quad a_4 = \frac{\alpha + M(c^2 - \alpha M) - 1}{M},$$
$$l_1 = \sqrt{\frac{2}{\beta}}\frac{d_S}{c^2}, \quad l_2 = \sqrt{\frac{2}{\beta}}\frac{(1+M)^2}{d_S M}, \quad l_3 = \frac{2(M^2-1)}{\beta c^2 M}. \tag{40}$$

The dispersion relation for the 2FP system is obtained by requiring the matrix determinant to be zero. This gives a normalized polynomial equation of order six in ω^2

$$\sum_{m=1}^{7}\sum_{n=1}^{5} A_{mn} k^{2(n-1)} \omega^{2(m-1)} = 0, \tag{41}$$

where the coefficients, $A_{mn} = A_{mn}(c, M, d_S, \alpha, \beta, \lambda)$, are tabulated in Appendix A. For each k^2, there are six two-fold degenerate waves. The degeneracy corresponds to the fact that we have six solutions for ω^2 such that we have for each solution of ω^2 two waves propagating in the opposite direction (one for $\omega > 0$ and the other corresponding to $\omega < 0$). The 2FP system is not a strictly hyperbolic system of partial differential equations: in such a system, we only encounter waves such that $\omega \propto k$. The 2FP system of wave equations includes dispersive waves where the dispersive waves stem from the electromagnetic source terms in the ion and electron momentum and energy equations.

An equivalent polynomial, although derived using different scaling, is given by Goedbloed and Poedts,[4] where the 6-wave structure associated with the two-fluid model and its asymptotic limits for extreme values of ω and k are discussed in terms of the background physical variables. We are particularly interested in three regions of these asymptotes under the current non-dimensionalization scheme, in order to facilitate a comparison against those of the other limiting forms of the two-fluid equations derived in Secs. III–VI. The readers are referred to Goedbloed and Poedts[4] for more details.

3. Asymptotic waves

First, the resonance limit, where $k \to \infty$ while keeping ω finite, is computed by solving the corresponding limit of Eq. (41), given by

$$\left(A_{51} + A_{52}\omega^2 + A_{53}\omega^4\right)k^8 = 0, \tag{42}$$

whose two positive roots lead to the ion and electron cyclotron resonant frequencies, ω_{ic} and ω_{ie}, respectively,

$$\omega_{ic} = \sqrt{\frac{2}{\beta}}\frac{\lambda}{d_S}\left(1 + \frac{1}{M}\right), \quad \omega_{ec} = \sqrt{\frac{2}{\beta}}\frac{\lambda}{d_S}(1+M). \tag{43}$$

These frequencies correspond to spatially localized cyclotron waves.

Second, the local high-frequency distinguished limit $k \to \infty$, $\omega \to \infty$, with ω/k finite, follows from the asymptotic dispersion equation as

$$A_{17}\omega^{12} + A_{26}k^2\omega^{10} + A_{35}k^4\omega^8 + A_{44}k^6\omega^6 + A_{53}k^8\omega^4 = 0, \tag{44}$$

which factorizes to give two repeated speed-of-light and the ion and electron sound speeds

$$\omega_{EM} = kc, \quad \omega_{is} = k\sqrt{(1-\alpha)\left(1+\frac{1}{M}\right)},$$
$$\omega_{es} = k\sqrt{\alpha(1+M)}. \tag{45}$$

The two EM waves are light waves with different polarization states. Under the present choice of u_{ref} in their dimensional form, these are

$$\tilde{\omega}_{EM} = \tilde{k}\tilde{c}, \quad \tilde{\omega}_{is} = \tilde{k}\sqrt{\frac{\gamma\tilde{p}_{i0}}{\tilde{n}_{i0}\tilde{m}_i}}, \quad \tilde{\omega}_{es} = \tilde{k}\sqrt{\frac{\gamma\tilde{p}_{e0}}{\tilde{n}_{e0}\tilde{m}_e}}. \tag{46}$$

In the limit $k \to 0$, the high frequency waves, viz., the two EM light waves, branch off and along with the electron acoustic wave asymptote to three plasma waves corresponding to the plasma frequency and the upper and lower cutoff plasma frequencies.[4] These high frequency waves, as will be seen later, are removed from the dispersion relation in the limit $c \to \infty$.

Finally, the global low-frequency limit $k \to 0$, $\omega \to 0$, with ω/k finite, is obtained from

$$A_{14}\omega^6 + A_{23}k^2\omega^4 + A_{32}k^4\omega^2 + A_{41}k^6 = 0, \tag{47}$$

which contains the perturbed MHD Alfvén wave and two acoustic waves due to finite speed-of-light, given by

$$\omega_A = k_\parallel\sqrt{\frac{2}{\beta + 2/c^2}}, \quad \omega_{f,s} = \left(\frac{k^2(2+\beta) + 2k_\parallel^2/c^2}{2(\beta+2/c^2)} \pm \sqrt{\frac{(1+\beta/2)^2k^4 - \left[\beta(2-1/c^2) + 2/c^2\right]k_\parallel^2k^2 + k_\parallel^4/c^4}{\beta+2/c^2)^2}}\right)^{1/2}. \tag{48}$$

One might immediately identify that these solutions converge exactly to the single-fluid MHD three waves (Alfvén, fast, and slow magnetosonic waves) in the limit as $c \to \infty$, which is discussed in more detail in Sec. VI B.

III. INFINITE SPEED-OF-LIGHT LIMIT

We first examine the $c \to \infty$ limit of the 2FP equations. This corresponds to the formal asymptotic limit where the small parameter $\delta_c \equiv 1/c^2 = u_{\mathrm{ref}}^2\mu_0\epsilon_0 \to 0$, while keeping

M, d_S, and β fixed. This is realized by requiring $(u_{\text{ref}}^2\epsilon_0) \to 0$ with μ_0 fixed.

A. Leading order equations: $c\to\infty$

1. Perturbation analysis

Since the $\delta_c \to 0$ limit is singular for the Ampére law (31) and for Gauss's divergence constraint on $\mathbf{E}$, (32), it is constructive to perform a perturbation expansion in powers of δ_c for all field variables of the form

$$\zeta = \zeta_0 + \zeta_1\delta_c + O(\delta_c^2), \tag{49}$$

where ζ is generically used to represent ρ, ρ_c, p, p_e, $\mathbf{u}$, $\mathbf{j}$, $\mathbf{B}$, and $\mathbf{E}$; the subscript zero now refers to the leading order solution; and the subscript one indicates first order correction. In particular, substituting the ρ_c and $\mathbf{E}$ expansions into (32) yields

$$O(1): \qquad \frac{\rho_{c_0}}{d_S}\sqrt{\frac{\beta}{2}} = \frac{e}{d_S}\sqrt{\frac{\beta}{2}}(n_{i_0} - n_{e_0}) = 0, \tag{50}$$

$$O(\delta_c): \qquad \frac{\rho_{c_1}}{d_S}\sqrt{\frac{\beta}{2}} = \frac{e}{d_S}\sqrt{\frac{\beta}{2}}(n_{i_1} - n_{e_1}) = \nabla\cdot\mathbf{E}_0, \tag{51}$$

where the definition for charge density is used. This implies that, in the limit of infinite speed-of-light ($c\to\infty$) while keeping β, $d_S > 0$, the plasma is exactly charge neutral, by $\rho_{c_0} = 0$, as the Gauss law for the electric field requires. However, for any large but finite speed-of-light ($\delta_c \ll 1$), the charge density does not need to vanish identically, and its value, being asymptotically small [$\sim O(\delta_c)$], is explicitly related to the leading order electric field solution $\mathbf{E}_0$, i.e.,

$$\rho_c = d_S\sqrt{\frac{2}{\beta}}(\nabla\cdot\mathbf{E}_0)\delta_c + O(\delta_c^2). \tag{52}$$

Consistency between the constraints (52) and (22), (30), which govern the time evolution of ρ_c and $\mathbf{E}$, respectively, must be ensured. Therefore, these two equations are also expanded, giving at order $O(1)$

$$\frac{\partial\rho_{c_0}}{\partial t} + \nabla\cdot\mathbf{j}_0 = 0, \tag{53}$$

$$\nabla\times\mathbf{B}_0 = \frac{\sqrt{\beta}}{\sqrt{2}d_S}\mathbf{j}_0 \tag{54}$$

and at order $O(\delta_c)$

$$\frac{\partial\rho_{c_1}}{\partial t} + \nabla\cdot\mathbf{j}_1 = 0, \tag{55}$$

$$\frac{\partial\mathbf{E}_0}{\partial t} - \nabla\times\mathbf{B}_1 = -\frac{\sqrt{\beta}}{\sqrt{2}d_S}\mathbf{j}_1. \tag{56}$$

Substituting the divergences of Eqs. (54) and (56) into (53) and (55), respectively, produces

$$\frac{\partial\rho_{c_0}}{\partial t} = 0, \quad \frac{\partial}{\partial t}\left(\rho_{c_1} - d_S\sqrt{\frac{2}{\beta}}(\nabla\cdot\mathbf{E}_0)\right) = 0. \tag{57}$$

It is clear that the divergence condition, (52), automatically satisfies the evolution equations for ρ_c to first order, and $\mathbf{E}$ to zeroth order, in δ_c.

2. Zeroth-order equations

To close the system, the remaining equations of motion must be included. Various flux and source terms involved in (21)–(33) simplify significantly owing to $\rho_{c_0} = 0$, giving

$$\mathbf{f}_{mon} = \frac{m_i^2}{e^2}\frac{\mathbf{j}_0\mathbf{j}_0}{M\rho_0} + O(\delta_c), \tag{58}$$

$$\mathbf{f}_{cur} = \frac{m_i}{e\rho_0 M}\mathbf{j}_0\mathbf{j}_0 + \frac{e(p_0 - (M+1)p_{e_0})}{m_i}\mathbf{I} + O(\delta_c), \tag{59}$$

$$\mathbf{s}_{cur} = \frac{\sqrt{2}(1-M)e}{\sqrt{\beta}d_S m_i}\mathbf{j}_0\times\mathbf{B}_0 + O(\delta_c), \tag{60}$$

$$\mathcal{E}_{erg} = \frac{m_i^2}{e^2}\frac{|\mathbf{j}_0|^2}{2M\rho_0} + O(\delta_c), \tag{61}$$

$$\begin{aligned}\mathbf{f}_{erg} &= \frac{|\mathbf{j}_0|^2 m_i^2}{2e^2 M\rho_0}\mathbf{u}_0 - \frac{\gamma m_i((M+1)p_{e_0} - p_0)}{(\gamma-1)eM\rho_0}\mathbf{j}_0\\ &\quad - \frac{|\mathbf{j}_0| m_i^2\left(|\mathbf{j}_0|(M-1)m_i - 2eM\rho_0|\mathbf{u}_0|\right)}{2e^3M^2\rho_0^2}\mathbf{j}_0 + O(\delta_c),\end{aligned} \tag{62}$$

$$S_{p_e} = O(\delta_c). \tag{63}$$

It is straightforward to collect the non-trivial leading order equations from the 2FP [see (21)–(33)] to obtain a closed set of equations for all of the zeroth order variables as $c\to\infty$, except for ρ_c, which is shown to be identically zero at the leading order. With the subscript zero dropped, this limiting set follows as

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{64}$$

$$\frac{\partial\rho\mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u} + p\mathbf{I}) = -\nabla\cdot\left(\frac{m_i^2}{e^2}\frac{\mathbf{j}\mathbf{j}}{M\rho}\right) + \frac{\sqrt{2}}{\sqrt{\beta}d_S}\mathbf{j}\times\mathbf{B}, \tag{65}$$

$$\begin{aligned}&\frac{\partial\mathbf{j}}{\partial t} + \nabla\cdot\left(\mathbf{u}\mathbf{j} + \mathbf{j}\mathbf{u} - \frac{m_i(1-M)}{e\rho M}\mathbf{j}\mathbf{j} + \frac{e(p-(M+1)p_e)}{m_i}\mathbf{I}\right)\\ &= \frac{\sqrt{2}e}{\sqrt{\beta}d_S m_i}\left(\frac{eM\rho}{m_i}(\mathbf{E} + \mathbf{u}\times\mathbf{B}) + (1-M)\mathbf{j}\times\mathbf{B}\right),\end{aligned} \tag{66}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\mathcal{E} + \frac{m_i^2}{e^2}\frac{|\mathbf{j}|^2}{2M\rho}\right) + \nabla\cdot\left((\mathcal{E}_h + p)\mathbf{u} + \frac{2}{\beta}\mathbf{E}\times\mathbf{B}\right)\\ &+ \nabla\cdot\mathbf{f}_{erg} = 0,\end{aligned} \tag{67}$$

$$\frac{\partial p_e}{\partial t} + \left(\mathbf{u} - \frac{m_i}{e\rho}\mathbf{j}\right)\cdot\nabla p_e + \gamma p_e\nabla\cdot\left(\mathbf{u} - \frac{m_i}{e\rho}\mathbf{j}\right) = 0, \tag{68}$$

$$\frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = 0, \tag{69}$$

$$\nabla\times\mathbf{B} = \frac{\sqrt{\beta}}{\sqrt{2}d_S}\mathbf{j}, \tag{70}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{71}$$

where

$$\mathcal{E} = \mathcal{E}_h + \frac{|\mathbf{B}|^2}{\beta}, \quad \mathcal{E}_h = \frac{p}{\gamma - 1} + \frac{1}{2}\rho|\mathbf{u}|^2, \tag{72}$$

$$\mathbf{f}_{erg} = \frac{|\mathbf{j}|^2 m_i^2}{2e^2 M\rho}\mathbf{u} - \frac{\gamma m_i((M+1)p_e\rho - p\rho)}{(\gamma-1)eM\rho^2}\mathbf{j} - \frac{|\mathbf{j}|m_i^2(|\mathbf{j}|(M-1)m_i - 2eM\rho|\mathbf{u}|)}{2e^3M^2\rho^2}\mathbf{j}. \tag{73}$$

We denote this system the two-fluid MHD equations, denoted henceforth as 2FMHD, for the strict charge neutrality imposed at zeroth order.

3. Discussion

The preceding perturbation analysis provides a clear interpretation of the "quasi-neutrality assumption" often discussed in magnetohydrodynamics.[5–7,11] Because $\rho_c \to 0$ as $c \to \infty$, charge separation does not occur in the limit. However, for finite but large c, charge separation exists at a magnitude which can be consistently determined using (52), i.e., the divergence constraint on the electric field (Gauss's law). Some authors[6,11] discussed this using scaling arguments in the form of a "quasi-neutrality approximation" for the single-fluid MHD equations. We show that in the $c \to \infty$ limit, strict charge neutrality holds in the limiting 2FMHD system because the divergence constraint only applies to first and higher order perturbations of ρ_c which do not contribute in any other zeroth order equations.

It follows from zeroth-order charge neutrality that the evolution of the electric field, $\mathbf{E}$, is locked with that of the magnetic field, $\mathbf{B}$, in the 2FMHD equations. To obtain $\mathbf{B}$, one uses Faraday's law (69) and eliminates both $\mathbf{j}$ and $\mathbf{E}$ by taking the curl of the current equation (66) after utilizing (70). $\mathbf{E}$ is then retrieved after $\mathbf{B}$, $\mathbf{u}$, and p_e are known, again using the current equation. It is also clear that the electron pressure decouples from the system, owing to the identity $\nabla \times (\nabla p_e) = 0$.

We remark that the plasma parameter β can be absorbed into $\mathbf{B}$ and $\mathbf{E}$ by defining

$$\bar{\mathbf{B}} = \frac{\mathbf{B}}{\sqrt{\beta}}, \quad \bar{\mathbf{E}} = \frac{\mathbf{E}}{\sqrt{\beta}}. \tag{74}$$

This also applies to the original non-dimensional 2FP equations (21)–(33). This is equivalent to removing the independent scale B_{ref} in Eq. (5) and replacing it by $(2\mu_0 n_{\mathrm{ref}} m_{\mathrm{ref}} u_{\mathrm{ref}}^2)^{1/2}$. As a result, Eqs. (64)–(71) expressed in terms of these rescaled magnetic and electric fields become independent of β, effectively by substituting $\beta = 1$ and writing $\bar{\mathbf{B}}$ and $\bar{\mathbf{E}}$ in places of $\mathbf{B}$ and $\mathbf{E}$, respectively, in the original equations. The β-independent equation set is computationally convenient since any strength of the external magnetic field can still be accommodated by suitable choice of initial/boundary conditions. The benefit of using the 2FMHD equations over the two-fluid model for numerical solutions is that the infinite speed-of-light limit eliminates fast transients that require stringently small time steps to resolve. However, as discussed above, the disadvantage of the 2FMHD equations is that the $\mathbf{E}$ field has to be solved implicitly through other variables, whose solutions depend on nested differential operators that are expensive to compute.

B. Dispersion relation for 2FMHD

1. Plane wave solutions

Next, we examine the behavior of waves admitted by the asymptotic two-fluid system as $c \to \infty$, considering again a homogeneous background. Using the same perturbations given in Eq. (35), the two-fluid system in this limit linearizes to

$$\begin{aligned}
&\frac{\partial \rho'}{\partial t} + \nabla\cdot\mathbf{u}' = 0, \quad \frac{\partial \mathbf{u}'}{\partial t} + \nabla p' = \frac{1}{d_S}\sqrt{\frac{2}{\beta}}\mathbf{j}'\times\mathbf{b},\\
&\frac{\partial \mathbf{j}'}{\partial t} + \frac{1+M}{M}\nabla p' - \frac{(1+M)^2}{M}\nabla p_e'\\
&\quad = \frac{1}{Md_S}\sqrt{\frac{2}{\beta}}\Big[(1+M)^2(\mathbf{E}' + \mathbf{u}'\times\mathbf{b}) + (1-M^2)\mathbf{j}'\times\mathbf{b}\Big],\\
&\frac{\partial p'}{\partial t} + \nabla\cdot\mathbf{u}' = 0, \quad \frac{\partial p_e'}{\partial t} + \alpha\nabla\cdot\mathbf{u}' = 0,\\
&\frac{\partial \mathbf{B}'}{\partial t} + \nabla\times\mathbf{E}' = 0, \quad \nabla\times\mathbf{B}' = \frac{1}{d_S}\sqrt{\frac{\beta}{2}}\mathbf{j}'.
\end{aligned} \tag{75}$$

In this case, it is convenient to eliminate $\mathbf{j}_1$ and $\mathbf{E}_1$ since p_{e1} does not enter the equations after the curl of $\mathbf{E}_1$ is taken. An application of the plane wave ansatz Eq. (37), together with Eq. (38), yields the following linear system in terms of $\mathbf{u}_1$ and $\mathbf{B}_1$:

$$\begin{bmatrix}
k_\perp^2 - \omega^2 & 0 & k_\perp k_\parallel & -\dfrac{2k_\parallel\omega}{\beta} & 0 & \dfrac{2k_\perp\omega}{\beta}\\
0 & \omega & 0 & 0 & \dfrac{2k_\parallel}{\beta} & 0\\
k_\perp k_\parallel & 0 & k_\parallel^2 - \omega^2 & 0 & 0 & 0\\
-k_\parallel & 0 & 0 & -\left(\dfrac{d_S^2 M k_\parallel^2}{(M+1)^2} + 1\right)\omega & i\sqrt{\dfrac{2}{\beta}}\dfrac{d_S(1-M)k_\parallel^2}{M+1} & \dfrac{d_S^2 M k_\perp k_\parallel\omega}{(M+1)^2}\\
0 & -k_\parallel & 0 & i\sqrt{\dfrac{2}{\beta}}\dfrac{d_S(M-1)k_\parallel^2}{M+1} & -\dfrac{\big(M(k^2 d_S^2 + M + 2) + 1\big)\omega}{(M+1)^2} & i\sqrt{\dfrac{2}{\beta}}\dfrac{d_S(1-M)k_\perp k_\parallel}{M+1}\\
k_\perp & 0 & 0 & \dfrac{d_S^2 M k_\perp k_\parallel\omega}{(M+1)^2} & i\sqrt{\dfrac{2}{\beta}}\dfrac{d_S(M-1)k_\perp k_\parallel}{M+1} & -\left(\dfrac{d_S^2 M k_\perp^2}{(M+1)^2} + 1\right)\omega
\end{bmatrix}
\begin{bmatrix}\mathbf{u}'\\ \\ \mathbf{B}'\end{bmatrix} = 0. \tag{76}$$

The corresponding wave dispersion relation follows from the matrix determinant, giving a polynomial equation

$$\sum_{m=1}^{4}\sum_{n=1}^{4} C_{mn}k^{2(m-1)}\omega^{2(n-1)} = 0, \tag{77}$$

where the coefficients C_{mn} form the following matrix:

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\dfrac{2\lambda^2+\beta+2}{\beta} & \dfrac{2d_S^2 M}{(M+1)^2} \\ 0 & \dfrac{4\lambda^2(\beta+1)}{\beta^2} & -\dfrac{2d_S^2\left(M^2\lambda^2+\lambda^2+M\left(-\lambda^2+\beta+1\right)\right)}{(M+1)^2\beta} & \dfrac{d_S^4M^2}{(M+1)^4} \\ -\dfrac{4\lambda^4}{\beta^2} & \dfrac{2d_S^2(M^2+1)\lambda^2}{(M+1)^2\beta} & -\dfrac{d_S^4M^2}{(M+1)^4} & 0 \end{bmatrix}. \tag{78}$$

It is straightforward to verify that one could arrive at the same result by directly taking the limit, $c\to\infty$, of the two-fluid dispersion relation derived in Eq. (41), after the renormalization where the coefficient A_{14} is made unity, i.e., $C_{mn}=\lim_{c\to\infty}(A_{mn}/A_{14})$. This provides a shortcut to obtain dispersion relations for other closed limiting systems to be derived in Secs. IV–VI.

2. Comparison with two-fluid plasma (2FP) waves

Evidently, compared to the original 2FP dispersion relation, Eq. (77) is now only of order three in both of ω^2 and k^2, suggesting that the cutoff and plasma frequencies occurring for $k\to 0$, $\omega>0$ are eliminated together with the double electromagnetic waves traveling at the speed-of-light due to charge neutrality. More is revealed by examining the high-frequency asymptote as $k\to\infty$, $\omega\to\infty$, that is, solving

$$C_{34}k^4\omega^6 + C_{43}k^6\omega^4 = 0, \tag{79}$$

showing that there is now only one positive root given by

$$\omega_{os} = k. \tag{80}$$

Therefore, compared to Eqs. (44) and (45), the electron and ion sound waves in the two-fluid system now coalesce into one overall sound wave, traveling at the speed of $u_{\mathrm{ref}}=\sqrt{\gamma\tilde{p}_0/\tilde{\rho}_0}$ in real units. This is caused by the linearization given in Eq. (75), where the electron pressure always remains a constant fraction, α, of the total pressure. Since ions and electrons also share the same number density in the $c\to\infty$ limit, their temperature ratio must stay the same at all time. Therefore, in the linear region, the two species described by the 2FMHD equations behave like a mixture, permitting only one sonic speed, and the difference between ω_{os} and $\omega_{is,es}$ will not decay when c is increased due to the limit being singular in the dispersion polynomial which decreases its order.

The low-frequency limit in this case follows from

$$C_{14}\omega^6 + C_{23}k^2\omega^4 + C_{32}k^4\omega^2 + C_{41}k^6 = 0 \tag{81}$$

and coincides with the continuous limit of Eqs. (47) and (48) as $c\to\infty$, giving exactly the well-known MHD 3-wave

$$\omega_A = k_\parallel\sqrt{\frac{2}{\beta}},$$

$$\omega_{f,s} = \left(\frac{k^2}{2}\left(\frac{\beta}{2}+1\right)\pm k\sqrt{\left[\frac{k}{2}\left(\frac{\beta}{2}+1\right)\right]^2-\frac{2k_\parallel^2}{\beta}}\right)^{1/2}. \tag{82}$$

More interestingly, the cyclotron resonance obtained from

$$\left(C_{41}+A_{42}\omega^2+A_{43}\omega^4\right)k^6 = 0 \tag{83}$$

is unaffected in the infinite speed-of-light limit, giving exactly the same ion and electron resonant frequencies as before in Eq. (43).

Figure 1 shows a direct comparison between the oblique waves of the 2FP system and the infinite speed-of-light 2FMHD system, where all positive roots of Eqs. (41) and (77) are plotted. The parameters (λ, α, β, d_S, and M) are chosen in accordance with the numerical example for a hydrogen plasma shown in Fig. 3.1 of Ref. 4. It is seen that indeed the infinite speed-of-light assumption leads to a loss of information about the high-frequency waves and a systematic departure from the ion or electron sound wave at high wave numbers, where sonic speeds for individual species merge into a combined value. Apart from this, the cyclotron resonance is retained exactly for both ions and electrons, and there is no noticeable deviation introduced by the $c\to\infty$ limit at low frequencies. We also note that the branch corresponding to the fast magnetosonic wave smoothly changes into the Whistler branch (with twice the slope on a log ω vs log k plot) before it asymptotes to the electron cyclotron wave.

IV. SMALL ELECTRON INERTIA LIMIT

Conventionally, a single-fluid plasma model refers to the approximation that electron inertia is negligible compared to the ion's. This limit is investigated in this section by applying $\delta_M\equiv 1/M\to 0$ to the two-fluid equations in the center-of-mass frame given in Eqs. (21)–(33), while keeping c, d_S, and β fixed.

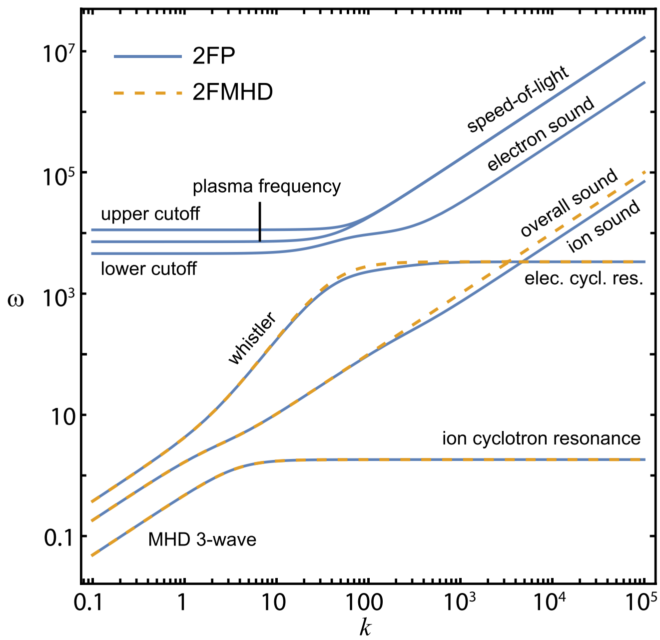

FIG. 1. Dispersion diagram for waves from an ideal two-fluid hydrogen plasma 2FP (solid lines) and its limit 2FMHD as $c \to \infty$ (dashed lines). $\lambda = 0.5$, $\alpha = 0.5$, $\beta = 0.15$, $d_S = 1$, and $M = 1836$. In particular, $c = 170$ for the two-fluid system.

A. Leading-order equations: $M \to \infty$

Similar to (49), a perturbation series here in powers of δ_M is used

$$\zeta = \zeta_0 + \zeta_1 \delta_M + O(\delta_M^2). \tag{84}$$

Conveniently, this limit only applies to the momentum (23), current (24), and energy equation (26), where the miscellaneous terms in Eqs. (25) and (27) simplify, giving

$$\begin{aligned}
\mathbf{f}_{mom} &= O(\delta_M), \quad \mathbf{f}_{cur} = \frac{-e p_{e_0}}{m_i \delta_M}\mathbf{I} + O(\delta_M^0),\\
\mathbf{s}_{cur} &= \frac{-\sqrt{2}\, e(\rho_{c_0}\mathbf{E}_0 + \mathbf{j}_0 \times \mathbf{B}_0)}{\sqrt{\beta} d_S m_i \delta_M} + O(\delta_M^0),\\
\mathcal{E}_{erg} &= \frac{|\mathbf{E}_0|^2}{\beta c^2} + O(\delta_M^2),\\
\mathbf{f}_{erg} &= -\frac{\gamma}{\gamma - 1}\frac{m_i p_{e_0}(\mathbf{j}_0 - \rho_{c_0}\mathbf{u}_0)}{e\rho_0 - \rho_{c_0} m_i} + O(\delta_M).
\end{aligned} \tag{85}$$

Therefore, under the expansion (84), collecting all the leading order equations from (21)–(33) gives another closed limiting system corresponding to the zero electron inertia limit at zeroth order. With the subscript zero removed, this system reads

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{86}$$

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot \mathbf{j} = 0, \tag{87}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + p\mathbf{I}) = \frac{\sqrt{2}}{\sqrt{\beta} d_S}(\mathbf{j} \times \mathbf{B} + \rho_c \mathbf{E}), \tag{88}$$

$$\nabla p_e = \frac{\sqrt{2}}{\sqrt{\beta} d_S}\left[-\frac{e\rho}{m_i}(\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \rho_c \mathbf{E} + \mathbf{j} \times \mathbf{B}\right], \tag{89}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\mathcal{E} + \frac{|\mathbf{E}_0|^2}{\beta c^2}\right) + \nabla \cdot \left((\mathcal{E}_h + p)\mathbf{u} + \frac{2}{\beta}\mathbf{E} \times \mathbf{B}\right)\\
&\quad + \nabla \cdot \mathbf{f}_{erg} = 0,
\end{aligned} \tag{90}$$

$$\frac{\partial p_e}{\partial t} + \left(\mathbf{u} - \frac{m_i}{e\rho}\mathbf{j}\right) \cdot \nabla p_e + \gamma p_e \nabla \cdot \left(\mathbf{u} - \frac{m_i}{e\rho}\mathbf{j}\right) + S_{p_e} = 0, \tag{91}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \tag{92}$$

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\frac{\sqrt{\beta}}{\sqrt{2} d_S}\mathbf{j}, \tag{93}$$

$$\frac{1}{c^2}\nabla \cdot \mathbf{E} = \frac{\sqrt{\beta}}{\sqrt{2} d_S}\rho_c, \tag{94}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{95}$$

where

$$\begin{aligned}
\mathcal{E} &= \mathcal{E}_h + \frac{|\mathbf{B}|^2}{\beta}, \quad \mathcal{E}_h = \frac{p}{\gamma - 1} + \frac{1}{2}\rho|\mathbf{u}|^2,\\
\mathbf{f}_{erg} &= -\frac{\gamma}{\gamma - 1}\frac{m_i p_e(\mathbf{j} - \rho_c \mathbf{u})}{e\rho - \rho_c m_i},
\end{aligned} \tag{96}$$

$$S_{p_e} = \frac{\rho_c m_i}{e\rho - \rho_c m_i}\left[\left(\mathbf{u} - \frac{m_i}{e\rho}\mathbf{j}\right) \cdot \nabla p_e + \gamma p_e \nabla \cdot \left(\mathbf{u} - \frac{m_i}{e\rho}\mathbf{j}\right)\right]. \tag{97}$$

Unlike the 2FMHD system [Eqs. (64)–(71)], here none of the variables vanish or decouple at zeroth order, and all the corresponding equations from the 2FP model are retained in the $M \to \infty$ limit. In particular, charge separation and the presence electromagnetic waves persist. For this reason, we refer to Eqs. (86)–(95) for the single- or one-fluid plasma model and henceforth denoted as 1FP.

One notable difference compared to the 2FP system is that the current equation (89) now loses its time derivative, resembling some key features of the generalized Ohm's law used in the Hall-MHD model (to be derived in Sec. VI A). In fact, the system formed by (86)–(95) extends the Hall-MHD model by incorporating the electrostatic component of the Lorentz force, $\rho_c \mathbf{E}$, in addition to the magnetic component, $\mathbf{j} \times \mathbf{B}$, into the momentum, current, and energy conservations. Owing to finite c, these two forces could be comparable in magnitude. Therefore, the 1FP description may also be viewed as an extension to the Hall-MHD model.

B. Dispersion relation for 1FP

1. Plane wave solutions

Since the structure of the 1FP system is that of the 2FP equations, plane waves subject to the zero electron inertia limit can be found following the same procedure described in Sec. II C, effectively turning each step into the corresponding $M \to \infty$ limit. The mass reference in this case is required to be $\tilde{m}_i$, such that $m_i = 1$, and (35) still holds. Without repeating details, the final expression for the dispersion relation of the 1FP model is given by

$$\sum_{m=1}^{5}\sum_{n=1}^{5} D_{mn} k^{2(n-1)} \omega^{2(m-1)} = 0, \tag{98}$$

where D_{mn} are explicitly tabulated in Appendix A. As mentioned in Sec. III B 1, these coefficients can also be accessed as $D_{mn} = \lim_{M\to\infty}(A_{mn}/A_{15})$ [see Eq. (41)].

2. Comparison with 2FP waves

By retaining finite c, it is expected that electromagnetic waves should be present. Indeed, solving the high-frequency asymptotic relation

$$D_{25}k^2\omega^8 + D_{34}k^4\omega^6 + D_{43}k^6\omega^4 + D_{52}k^8\omega^2 = 0 \tag{99}$$

produces two positive wave speeds

$$\omega_{EM} = kc, \quad \omega_{is} = k\sqrt{1-\alpha}, \tag{100}$$

for light and ion sound, respectively. Compared to (45), although the M-dependence is eliminated, the ion sound is still exact for our adjusted mass reference ($m_i = 1$). Because the electron is massless, its sonic speed escapes to infinity. Similarly, the cyclotron resonance now occurs only for ions, obtained from

$$D_{51}k^8 + D_{52}k^6\omega^2 = 0, \tag{101}$$

as

$$\omega_{ic} = \sqrt{\frac{2}{\beta}\frac{\lambda}{d_S}}. \tag{102}$$

A further consequence of the $M\to\infty$, $c<\infty$ limit lies in a reduced set of cut-off frequencies. Solving

$$D_{14}\omega^6 + D_{15}\omega^8 = 0 \tag{103}$$

gives a single cut-off frequency

$$\omega_{cut} = \frac{2+\beta c^2}{\sqrt{2\beta}\,d_S} = \frac{\omega_{ic}}{\lambda} + \frac{\lambda\omega_{pi}^2}{\omega_{ic}}. \tag{104}$$

This is neither the large M limit of the plasma frequency nor upper/lower cutoffs observed in the original 2FP model but is related to the ion plasma frequency ω_{pi}.[5] At the low-frequency end, the 1FP model exhibits identical behavior and has the waves given in (48), where finite c corrects the MHD 3-wave, independent of M.

A numerical comparison between the 2FP and the 1FP systems is shown in Fig. 2, where the same plasma considered in Fig. 1 is used.

It is seen that since the electron wave is lost, the whistler wave is forced to merge with the light wave at high frequencies, whereas the ion acoustic and resonant waves are preserved exactly. The single cutoff associated with the 1FP model should not be confused with the plasma frequency of the 2FP system because these two do not communicate through a continuous limit.

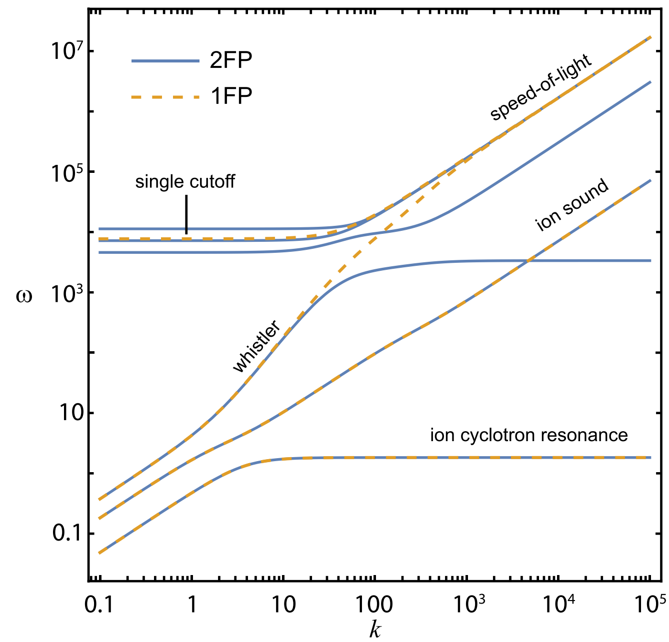

FIG. 2. Dispersion diagram for waves from an ideal two-fluid hydrogen plasma 2FP (solid lines) and its limit 1FP as $M\to\infty$ (dashed lines). $\lambda = 0.5$, $\alpha = 0.5$, $\beta = 0.15$, $d_S = 1$, and $c = 170$. In particular, $M = 1836$ for the two-fluid system.

V. ZERO SKIN DEPTH LIMIT

There are three commonly used assumptions in magnetohydrodynamics, often described by the celebrated single-fluid ideal MHD equations:[6] (a) charge quasi-neutrality, (b) negligible electron inertia, and (c) small Larmor radius. Presently, approximations (a) and (b) have been studied individually, first by applying the formal limit of $c\to\infty$ in Sec. III, and independently, $M\to\infty$ in Sec. IV, to the ideal 2FP equations. In this section, we isolate assumption (c), $d_L\to 0$, and revisit the concept of "quasi-neutrality" discussed in Sec. III A. Under the present non-dimensionlization scheme, the zero Larmor radius limit could be achieved by two means due to Eq. (17): either letting the plasma beta $\beta\to 0$ while keeping the d_S fixed or requiring $d_S\to 0$ while keeping β fixed. We use the latter limit applied directly to the 2FP equations (21)–(33), while keeping c, M, and β fixed. A brief discussion on the former route ($\beta\to 0$) is found in Appendix B.

A. Leading order equations: $d_S\to 0$

1. Perturbation analysis

In the 2FP model [Eqs. (21)–(33)], field variables ρ, ρ_c, p, p_e, $\mathbf{u}$, $\mathbf{j}$, $\mathbf{B}$, and $\mathbf{E}$ are expanded in powers of d_S, and this time in the form of

$$\zeta = \zeta_0 + \zeta_1 d_S + O(d_S^2), \tag{105}$$

where the subscripts indicate zeroth and first order quantities. It is most revealing to expand (31) and (32) first, showing that at order $O(d_S^{-1})$

$$\mathbf{j}_0 = \mathbf{0}, \tag{106}$$

$$\rho_{c_0} = 0 \tag{107}$$

and at order $O(d_S^0)$

$$\frac{1}{c^2}\frac{\partial \mathbf{E}_0}{\partial t} - \nabla \times \mathbf{B}_0 = -\frac{\sqrt{\beta}}{\sqrt{2}}\mathbf{j}_1, \tag{108}$$

$$\frac{1}{c^2}\nabla \cdot \mathbf{E}_0 = \frac{\sqrt{\beta}}{\sqrt{2}}\rho_{c_1}. \tag{109}$$

It follows immediately that the quasi-neutrality equation previously derived in (52) is here written alternatively with $d_S \to 0$ as

$$\rho_c = \frac{d_S}{c^2}\sqrt{\frac{2}{\beta}}(\nabla \cdot \mathbf{E}_0) + O(d_S^2). \tag{110}$$

Also, compatibility with the evolution equation at first order [expanded using (105)] is established by substituting explicit expressions for $\mathbf{j}_1$ and ρ_{c_1}, given in (108) and (109), into the charge density continuity equation (22). As a result, (22) can be safely removed from the $d_S \to 0$ limit.

Using Eqs. (105)–(107), the continuity (21) and momentum (23) equations jointly give at $O(d_S^0)$

$$\rho_0\left(\frac{\partial \mathbf{u}_0}{\partial t} + \mathbf{u}_0 \cdot \nabla \mathbf{u}_0\right) = -\nabla p_0 + \sqrt{\frac{2}{\beta}}(\mathbf{j}_1 \times \mathbf{B}_0 + \rho_{c_1}\mathbf{E}_0), \tag{111}$$

which now involves first order perturbations $\mathbf{j}_1$ and ρ_{c_1}. Fortunately, these two unknowns can be consistently eliminated using (108) and (109), respectively. Furthermore, the current equation (24) at $O(d_S^{-1})$ leads to

$$\mathbf{E}_0 + \mathbf{u}_0 \times \mathbf{B}_0 = 0, \tag{112}$$

which is precisely the Ohm law used in the single-fluid ideal MHD equations. It is stressed that we arrive at this result without invoking the $c \to \infty$ limit.

Similarly, the energy equation (26) at $O(d_S^0)$ gives

$$\frac{\partial}{\partial t}\left(\mathcal{E}_{h_0} + \frac{|\mathbf{B}_0|^2}{\beta} + \frac{|\mathbf{E}_0|^2}{\beta c^2}\right) + \nabla \cdot \left((\mathcal{E}_{h_0} + p_0)\mathbf{u}_0 + \frac{2}{\beta}\mathbf{E}_0 \times \mathbf{B}_0\right) = 0, \tag{113}$$

where

$$\mathcal{E}_{h_0} = \frac{p_0}{\gamma - 1} + \frac{1}{2}\rho_0|\mathbf{u}_0|^2. \tag{114}$$

Subtracting the kinetic, magnetic, and electric energies away from this total energy conservation leads to an equivalent equation for total pressure

$$\frac{\partial p_0}{\partial t} + \mathbf{u}_0 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \mathbf{u}_0 = 0. \tag{115}$$

The leading order equation for electron pressure obtained from expanding (28) shares the same operator on total pressure, that is,

$$\frac{\partial p_{e_0}}{\partial t} + \mathbf{u}_0 \cdot \nabla p_{e_0} + \gamma p_{e_0}\nabla \cdot \mathbf{u}_0 = 0. \tag{116}$$

Since p_{e_0} decouples from the general system, it can be omitted in the $d_S \to 0$ limiting set.

Combing these results, including those for (21) and (30), with (116) as an auxiliary relation, a significantly simplified system is obtained by only applying the $d_S \to 0$ limit to the 2FP model

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \mathbf{u}_0) = 0, \tag{117}$$

$$\rho_0\left(\frac{\partial \mathbf{u}_0}{\partial t} + \mathbf{u}_0 \cdot \nabla \mathbf{u}_0\right) = -\nabla p_0 + \sqrt{\frac{2}{\beta}}(\mathbf{j}_1 \times \mathbf{B}_0 + \rho_{c_1}\mathbf{E}_0), \tag{118}$$

$$\frac{\partial p_0}{\partial t} + \mathbf{u}_0 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \mathbf{u}_0 = 0, \tag{119}$$

$$\mathbf{E}_0 + \mathbf{u}_0 \times \mathbf{B}_0 = \mathbf{0}, \tag{120}$$

$$\frac{\partial \mathbf{B}_0}{\partial t} + \nabla \times \mathbf{E}_0 = \mathbf{0}, \tag{121}$$

$$\frac{1}{c^2}\frac{\partial \mathbf{E}_0}{\partial t} - \nabla \times \mathbf{B}_0 = -\sqrt{\frac{\beta}{2}}\mathbf{j}_1, \tag{122}$$

$$\frac{1}{c^2}\nabla \cdot \mathbf{E}_0 = \sqrt{\frac{\beta}{2}}\rho_{c_1}, \tag{123}$$

$$\nabla \cdot \mathbf{B}_0 = 0. \tag{124}$$

We call this system the quasi-neutral MHD equations (QMHD) for reasons discussed below.

2. Discussion

Owing to simplifications made by (106) and (107), all of the leading order equations are independent of M in the $d_S \to 0$ limit. This does not imply that the QMHD is a single-fluid model. In fact for any $M > 0$, by solving the augmented system which includes (116) for the electron pressure, all field variables for individual species can be recovered via the transformation given in (19). Since ion and electron pressures are governed by the same equation, they behave thermally like a mixture.

That first order perturbations, namely ρ_{c_1} and $\mathbf{j}_1$, appear in the leading order system which is very different from the 2FMHD and 1FP models. An important consequence is that the quasi-neutral effect originating from the $\nabla \cdot \mathbf{E}_0$ constraint contributes explicitly in the QMHD model while being neglected as a next order correction in the 2FMHD system, where strict neutrality applies [see the derivation for (64)–(71) in Sec. III A]. Here, because of finite c, charge separation at first order in d_S gives rise to an electrostatic force that is not negligible, and the displacement current is important in determining the first order current $\mathbf{j}_1$.

Finally, without invoking either $c \to \infty$ or $M \to \infty$, we arrive at a closed system that is remarkably similar to the single-fluid ideal MHD model. In fact, further applying the $\delta_c = 1/c^2 \to 0$ limit to the QMHD system apparently leads to

$$\rho_{c_1} = O(\delta_c), \tag{125}$$

$$\nabla \times \mathbf{B}_0 - \sqrt{\frac{2}{\beta}}\mathbf{j}_1 = O(\delta_c). \tag{126}$$

Thus, ρ_{c_1} exits the system together with the $\nabla \cdot \mathbf{E}$ constraint, eliminating the displacement current and electrostatic force. The ideal MHD equations are hence obtained exactly at order unity. This argument can be made rigorous by a perturbation analysis on the QMHD system, following a similar procedure given in Sec. III A.

B. Dispersion relation for QMHD

Plane waves associated with the QMHD system, (117)–(124), are found for the same homogeneous background considered in Sec. II C. Applying the ansatz given in (35), (37), and (38) again to the linearized QMHD system leads to the following eigenvalue problem in terms of $\mathbf{u}'$ and $\mathbf{B}'$

$$\begin{bmatrix} k_\perp^2 - \left(1 + \frac{2}{c^2\beta}\right)\omega^2 & 0 & k_\perp k_\parallel & -\frac{2}{\beta}k_\parallel\omega & 0 & \frac{2}{\beta}k_\perp\omega \\ 0 & -\left(1+\frac{2}{c^2\beta}\right)\omega & 0 & 0 & -\frac{2k_\parallel}{\beta} & 0 \\ k_\perp k_\parallel & 0 & k_\parallel^2 - \omega^2 & 0 & 0 & 0 \\ -k_\parallel & 0 & 0 & -\omega & 0 & 0 \\ 0 & -k_\parallel & 0 & 0 & -\omega & 0 \\ k_\perp & 0 & 0 & 0 & 0 & -\omega \end{bmatrix} \begin{bmatrix} \mathbf{u}' \\ \\ \mathbf{B}' \end{bmatrix} = 0. \tag{127}$$

The dispersion relation follows from the determinant, giving

$$\left[\left(1+\frac{2}{\beta c^2}\right)\omega^2 - \frac{2k_\parallel^2}{\beta}\right]\left[\left(1+\frac{2}{\beta c^2}\right)\omega^4 - \left(\frac{2+\beta}{\beta} + \frac{2\lambda^2}{\beta c^2}\right)k^2\omega^2 + \frac{2k_\parallel^2 k^2}{\beta}\right] = 0. \tag{128}$$

Once again, it is verified that this relation is also directly obtained from Eq. (41) as

$$\lim_{d_S \to 0} \sum_{m,n} \frac{A_{mn}}{A_{14}} k^{2n-2}\omega^{2m-2} = 0. \tag{129}$$

It can be safely concluded that all three limits of the 2FP model considered so far admit corresponding limiting dispersion relations, after the coefficient of a unique term in (41) is appropriately renormalized to unity for the limits to exist. Interestingly, (128) is also identical to the 2FP low-frequency asymptote given in (47), whose solutions are already shown in Eq. (48). It is not surprising that the QMHD waves provide finite speed-of-light corrections to the ideal MHD three waves. Curiously, despite finite c, electromagnetic waves traveling at the speed-of-light are lost in the QMHD system. This is because the electric field $\mathbf{E}_0$ is no longer an independent variable due to the ideal Ohm law (120) with the result that the Ampére law (122) serves as an explicit expression for the current perturbation $\mathbf{j}_1$ in terms of the magnetic field $\mathbf{B}_0$.

VI. MAGNETOHYDRODYNAMIC REDUCTIONS

Having independently investigated the $c \to \infty$, $M \to \infty$, and $d_S \to 0$ limits in Secs. III, IV, and V, respectively, we now investigate limiting forms of the 2FP model subject to multiple limits. This is achieved under the present framework by applying consecutive limits in terms of c, M, and d_S, while exhausting all permutations, should any two of the limits do not commute. Fortunately, it is not difficult to verify that all three limits do commute, resulting in two more sets of equations that are widely used in magnetohydrodynamics, namely, the single-fluid Hall-MHD and ideal MHD equations. In this section, we demonstrate the derivation following one path for each of these two models without a full proof of the commutative property.

A. Hall-MHD reduction

First, we utilize the $c \to \infty$ limit of the 2FP model (2FMHD) and apply additionally the zero electron mass limit $\delta_M = 1/M \to 0$. The plasma skin depth, d_S, and hence, Larmor radius, d_L [see (14)], are held finite.

1. Leading order equations: $c \to \infty$, $M \to \infty$

Using perturbation expansions in the form given by Eq. (84), at order $O(\delta_M^0)$, the 2FMHD continuity (64) and momentum (65) equations jointly give

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \frac{\sqrt{2}}{\sqrt{\beta}d_S}\mathbf{j}\times\mathbf{B} + O(\delta_M), \tag{130}$$

while the energy equation (67) leads to

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left((\mathcal{E}_h + p)\mathbf{u} + \frac{2}{\beta}\mathbf{E}\times\mathbf{B} - \frac{\gamma}{\gamma-1}\frac{m_i p_e}{e\rho}\mathbf{j}\right) = O(\delta_M), \tag{131}$$

where

$$\mathcal{E}_h = \frac{p}{\gamma - 1} + \frac{1}{2}\rho|\mathbf{u}|^2, \quad \mathcal{E} = \mathcal{E}_h + \frac{|\mathbf{B}|^2}{\beta}. \tag{132}$$

The current equation (66) at order $O(\delta_M^{-1})$ reduces to

$$\frac{\sqrt{2}}{\sqrt{\beta}d_S}\left(\frac{e\rho}{m_i}(\mathbf{E} + \mathbf{u}\times\mathbf{B}) - \mathbf{j}\times\mathbf{B}\right) + \nabla p_e = O(\delta_M). \tag{133}$$

Recalling $n_i - n_e = O(\delta_c)$ from (52) and $m_e/m_i = \delta_M$ by definition, (133) can be shown to be asymptotically equivalent to the generalized Ohm's law[11] with zero resistivity, namely,

$$\frac{\sqrt{2}}{\sqrt{\beta}d_S}\left(\mathbf{E} + \mathbf{u}\times\mathbf{B} - \frac{1}{e\,n_e}\mathbf{j}\times\mathbf{B}\right) + \frac{1}{e\,n_e}\nabla p_e = O(\delta_c) + O(\delta_M). \tag{134}$$

Combined with the leading-order equations in the 2FMHD system, a complete set of single-fluid equations, in the $\delta_c \to 0$, $\delta_M \to 0$ limit, is given by

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{135}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \frac{\sqrt{2}}{\sqrt{\beta}d_S}\mathbf{j}\times\mathbf{B}, \tag{136}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla\cdot\left((\mathcal{E}_h + p)\mathbf{u} + \frac{2}{\beta}\mathbf{E}\times\mathbf{B} - \frac{\gamma}{\gamma-1}\frac{m_i p_e}{e\rho}\mathbf{j}\right) = 0, \tag{137}$$

$$\frac{\partial p_e}{\partial t} + \left(\mathbf{u} - \frac{m_i}{e\rho}\mathbf{j}\right)\cdot\nabla p_e + \gamma p_e \nabla\cdot\left(\mathbf{u} - \frac{m_i}{e\rho}\mathbf{j}\right) = 0, \tag{138}$$

$$\frac{\sqrt{2}}{\sqrt{\beta}d_S}\left(\mathbf{E} + \mathbf{u}\times\mathbf{B} - \frac{1}{e\,n_e}\mathbf{j}\times\mathbf{B}\right) + \frac{1}{e\,n_e}\nabla p_e = 0, \tag{139}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = \mathbf{0}, \tag{140}$$

$$\nabla\times\mathbf{B} = \frac{\sqrt{\beta}}{\sqrt{2}d_S}\mathbf{j}, \tag{141}$$

$$\nabla\cdot\mathbf{B} = 0. \tag{142}$$

After converting to dimensional quantities, this equation set can be identified precisely as a more complete version of the Hall-MHD model (see the study by Srinivasan and Shumlak,[5] Hameiri,[12] and Hagstrom and Hameiri[13]), obtained by both the $c \to \infty$, $M \to \infty$ limits. The electron thermal term, ∇p_e, in (134) is sometimes dropped in the Hall-MHD model,[7–9] which is a choice made for simplicity typically based on physical arguments.[11]

2. Dispersion relation for Hall-MHD

Waves admitted by the Hall-MHD system given in (135)–(142) can be routinely determined in the linear region using the Fourier ansatz (37). Here, we take the established shortcut, by letting $M \to \infty$ in (77) [or $c \to \infty$ in (98)], to yield

$$\left(\omega^2 - \frac{2k_\parallel^2}{\beta}\right)\left(\omega^4 - \left(\frac{2}{\beta} + 1\right)k^2\omega^2 + \frac{2k_\parallel^2 k^2}{\beta}\right) - \frac{2d_S^2}{\beta}k_\parallel^2 k^2\omega^2(\omega^2 - k^2) = 0. \tag{143}$$

In its dimensional form, this relation is well known.[8,9] Asymptotic solutions to (143) feature the same overall sound and low-frequency waves as those observed in the 2FMHD system [see (80) and (82)]. However, these differ from the 1FP model where the sonic speed of ions is retained, and the MHD 3-wave depends on finite c. A unique property of the Hall-MHD dispersion relation is that its Whistler wave frequency is now unbounded and grows quadratically with $k \to \infty$ as

$$\omega = \sqrt{\frac{2}{\beta}}d_S k_\parallel k. \tag{144}$$

This is different from the 2FMHD system where the Whistler wave levels off at the electron cyclotron resonance and the 1FP model where it merges with the speed-of-light.

These results are illustrated in Fig. 3, where the entire Hall-MHD dispersion diagram is compared against that of the 2FMHD system in (a) and the 1FP model in (b). Clearly, the unbounded growth of the Whistler wave speed with k, also observed by Srinivasan and Shumlak,[5] makes the Hall-MHD model distinct, as a double limit ($c \to \infty$, $M \to \infty$).

B. Ideal MHD reduction

It was noted in Sec. V A 2 that the celebrated single-fluid ideal MHD equations can be derived from the $c \to \infty$ limit in addition to the $d_S \to 0$ condition to the 2FP model, without restrictions on the ion or electron masses (finite M). Here, as an example of the limits being commutative, we interchange the limiting order and formally arrive at the ideal MHD system via taking the $d_S \to 0$ limit of the 2FMHD equations.

1. Leading-order equations: $c\to\infty$, $d_S\to 0$

The d_S perturbation expansions used in (105) are employed. For completeness, the expanded 2FMHD equations (64)–(71) are the following: continuity equation

$$O(1): \frac{\partial \rho_0}{\partial t} + \nabla\cdot(\rho_0\mathbf{u}_0) = 0; \tag{145}$$

momentum equation

$$O(d_S^{-1}): \mathbf{j}_0\times\mathbf{B}_0 = 0, \tag{146}$$

$$O(d_S^0): \rho_0\left(\frac{\partial \mathbf{u}_0}{\partial t} + \mathbf{u}_0\cdot\nabla\mathbf{u}_0\right) + \nabla p_0 = -\nabla\cdot\left(\frac{m_i^2}{e^2}\frac{\mathbf{j}_0\mathbf{j}_0}{M\rho_0}\right) + \sqrt{\frac{2}{\beta}}(\mathbf{j}_0\times\mathbf{B}_1 + \mathbf{j}_1\times\mathbf{B}_0); \tag{147}$$

current equation

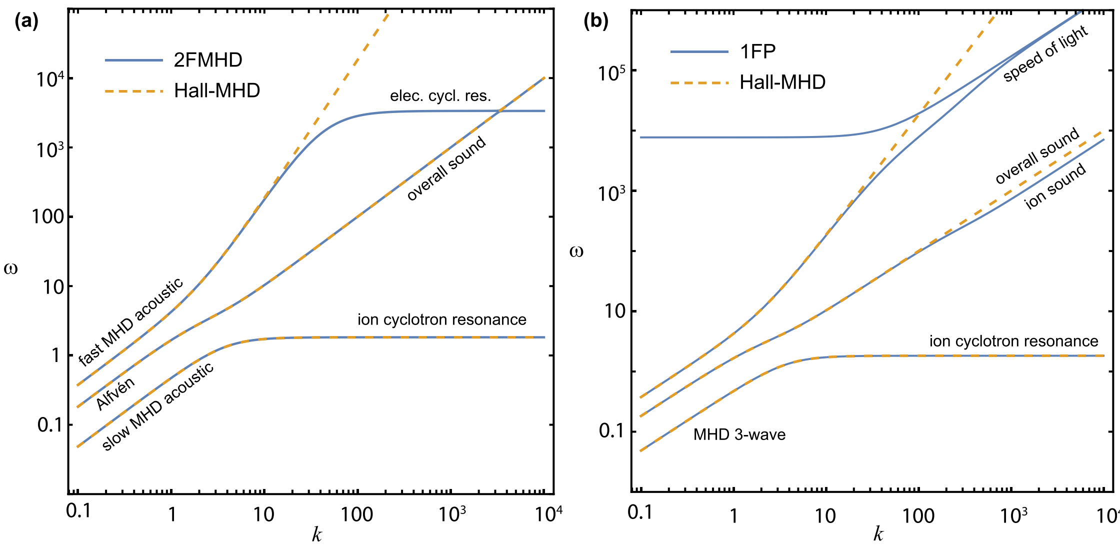

FIG. 3. Wave comparison of a hydrogen plasma described by the Hall-MHD equations against (a) the 2FMHD model where $M = 1836$ and (b) the 1FP model, where $c = 170$. In both cases, $\lambda = 0.5$, $\beta = 0.15$, and $d_S = 1$.

$$O(d_S^{-1}) : \frac{eM\rho_0}{m_i}(\mathbf{E}_0 + \mathbf{u}_0 \times \mathbf{B}_0) + (1 - M)\mathbf{j}_0 \times \mathbf{B}_0 = \mathbf{0}; \tag{148}$$

energy equation

$$O(d_S^0) : \frac{\partial}{\partial t}\left(\mathcal{E}_0 + \frac{m_i^2}{e^2}\frac{|\mathbf{j}_0|^2}{2M\rho_0}\right) + \nabla \cdot \left((\mathcal{E}_{h_0} + p_0)\mathbf{u} + \frac{2}{\beta}\mathbf{E}_0 \times \mathbf{B}_0\right) + \nabla \cdot \mathbf{f}_{ergo} = 0, \tag{149}$$

where

$$\mathcal{E}_{h_0} = \frac{p_0}{\gamma - 1} + \frac{1}{2}\rho_0|\mathbf{u}_0|^2, \quad \mathcal{E}_0 = \mathcal{E}_{h_0} + \frac{|\mathbf{B}_0|^2}{\beta},$$
$$\mathbf{f}_{ergo} = \frac{|\mathbf{j}_0|^2 m_i^2}{2e^2 M\rho_0}\mathbf{u}_0 - \frac{\gamma m_i((M+1)p_{e0} - p_0)}{(\gamma - 1)eM\rho_0}\mathbf{j}_0 - \frac{|\mathbf{j}_0|m_i^2\left(|\mathbf{j}_0|(M-1)m_i - 2eM\rho_0|\mathbf{u}_0|\right)}{2e^3M^2\rho_0^2}\mathbf{j}_0; \tag{150}$$

electron pressure equation

$$O(d_S^0) : \frac{\partial p_{e_0}}{\partial t} + \left(\mathbf{u}_0 - \frac{m_i}{e\rho_0}\mathbf{j}_0\right) \cdot \nabla p_{e_0} + \gamma p_{e_0}\nabla \cdot \left(\mathbf{u}_0 - \frac{m_i}{e\rho_0}\mathbf{j}_0\right) = 0; \tag{151}$$

and the reduced Maxwell equations

$$O(d_S^{-1}) : \mathbf{j}_0 = \mathbf{0}, \tag{152}$$

$$O(d_S^0) : \nabla \times \mathbf{B}_0 = \sqrt{\frac{\beta}{2}}\mathbf{j}_1, \tag{153}$$

$$O(d_S^0) : \frac{\partial \mathbf{B}_0}{\partial t} + \nabla \times \mathbf{E}_0 = \mathbf{0}, \tag{154}$$

$$O(d_S^0) : \nabla \cdot \mathbf{B}_0 = 0. \tag{155}$$

Since the lowest order Ampere's law again forces $\mathbf{j}_0 = \mathbf{0}$, the M dependence is eliminated in all of leading order equations, yielding the closed system given by

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \mathbf{u}_0) = 0, \tag{156}$$

$$\rho_0\left(\frac{\partial \mathbf{u}_0}{\partial t} + \mathbf{u}_0 \cdot \nabla \mathbf{u}_0\right) = -\nabla p_0 + \sqrt{\frac{2}{\beta}}\mathbf{j}_1 \times \mathbf{B}_0, \tag{157}$$

$$\frac{\partial}{\partial t}\left(\mathcal{E}_{h_0} + \frac{|\mathbf{B}_0|^2}{\beta}\right) + \nabla \cdot \left((\mathcal{E}_{h_0} + p_0)\mathbf{u}_0 + \frac{2}{\beta}\mathbf{E}_0 \times \mathbf{B}_0\right) = 0, \tag{158}$$

$$\mathbf{E}_0 + \mathbf{u}_0 \times \mathbf{B}_0 = \mathbf{0}, \tag{159}$$

$$\frac{\partial \mathbf{B}_0}{\partial t} + \nabla \times \mathbf{E}_0 = \mathbf{0}, \tag{160}$$

$$\nabla \times \mathbf{B}_0 = \sqrt{\frac{\beta}{2}}\mathbf{j}_1, \tag{161}$$

$$\nabla \cdot \mathbf{B}_0 = 0, \tag{162}$$

where

$$\mathcal{E}_{h_0} = \frac{p_0}{\gamma - 1} + \frac{1}{2}\rho_0|\mathbf{u}_0|^2. \tag{163}$$

As in the QMHD system [see (117)–(124)], the electron pressure, although decoupled, is governed by the same equation for total pressure, obtained from the energy conservation and given by

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_0 \cdot \nabla + \gamma \nabla \cdot \mathbf{u}_0\right)p_0 = 0. \tag{164}$$

That is, by utilizing $c \to \infty$ and $d_S \to 0$ alone, one arrives at the ideal MHD equations (156)–(162), where the single-fluid assumption made by $M \to \infty$ is not necessary. This finding suggests that a plasma with comparable ion and electron

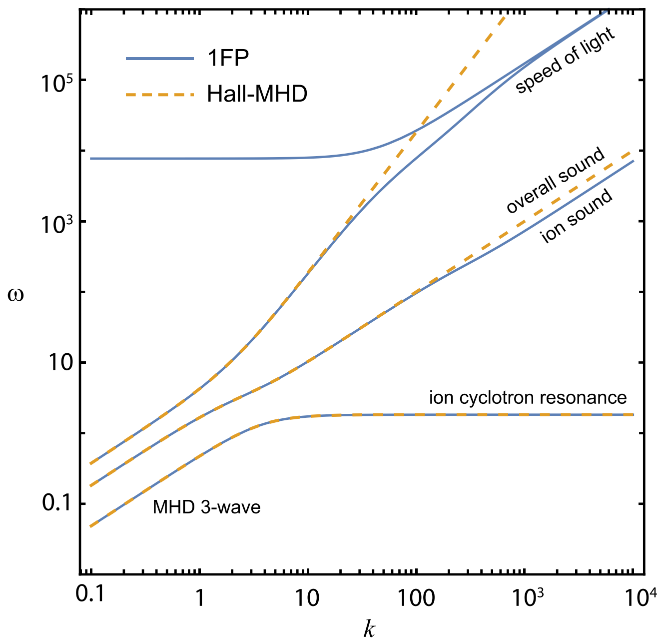

FIG. 4. Dispersion diagram for waves of a hydrogen plasma described by the ideal MHD (solid lines), Hall-MHD (dashed line), and QMHD equations (circles). $\lambda = 0.5$, $\beta = 0.15$, and $d_S = 1$ for Hall-MHD and $c = 10$ for QMHD.

masses can still be well described by the ideal MHD equations in the center-of-mass frame, assuming that c is large and d_S is small.

The behavior of the MHD system is now controlled by a single parameter β, which measures the relative magnitude of the thermal energy to the magnetic energy. The $\beta \to 0$ limit, outlined briefly in Appendix B, corresponds to a description of the overwhelming background magnetic field, whereas the $\beta \to \infty$ limit is the Euler equation for a fluid subject to no body force.

2. Dispersion relation for ideal MHD equations

The MHD three-waves are classic results in plasma physics. Their dispersion relation is found by taking the $d_S \to 0$ limit of (77), or equivalently the $c \to \infty$ limit of (128), giving

$$\left(\omega^2 - \frac{2k_\parallel^2}{\beta}\right)\left(\omega^4 - \left(\frac{2}{\beta}+1\right)k^2\omega^2 + \frac{2k_\parallel^2 k^2}{\beta}\right) = 0, \quad (165)$$

whose solutions are already shown in Eq. (82). This clarifies that the MHD equations are a reasonably accurate model for low frequency, macroscopic ideal plasma processes. Compared to the QMHD model, where the global three waves are exact, the ideal MHD model introduces deviations from the 2FP system in this region which decay rapidly at a rate of $O(1/c^2)$ [see (48)].

Both the QMHD and ideal MHD systems forget all information at high frequencies and thus fail to capture the ion cyclotron resonance and the overall sound wave occurring at large wave numbers that the Hall-MHD model is able to maintain, as shown in Fig. 4.

Incorporating the infinite speed-of-light assumption, both the Hall-HMD and ideal MHD equations inherit the property of strict neutrality from the 2FMHD model in the limit, where first order charge separation does not affect the models at leading order. In particular, for the ideal MHD system, where the skin depth is also made small, the magnitude of the quasi-neutral effect is of order $O(d_S/c^2)$, asymptotically smaller than any other liming models that are derived in the paper.

VII. SUMMARY AND CONCLUSIONS

By exploring various limits with respect to the speed-of-light, c, mass ratio, M, and plasma skin depth, d_S, five different limiting forms of the two-fluids plasma (2FP) equations are derived, namely, (a) the 2FMHD equations, (b) the 1FP equations, (c) the QMHD equations, (d) the Hall-MHD equations, and (e) the ideal MHD equations. For all the derived systems, their corresponding dispersion relations are also analytically determined and compared. Table I summarizes the key results.

The hierarchy of closed systems listed in Table I documents how plasmas can be appropriately modeled in situations where any combination of the limits investigated is appropriate. This may be particularly valuable for problems where only one of the limits applies, which lie in the parameter space in between where the two-fluid plasma and Hall-MHD models are appropriate. The first of these systems, the 2FMHD equations, is the zeroth-order description in the $c \to \infty$ limit. Strict charge neutrality holds in the limiting 2FMHD equations, but it nonetheless uniquely determines the perturbation charge non-neutrality at first order for large but finite c. The electron pressure decouples from the system, and information on high frequency waves is lost. The second system, the 1FP equations, corresponds to the $M \to \infty$ limit. In this system, no variables decouple from the system, and the presence of charge separation and electromagnetic waves persists. The evolution equation for the current does lose its time derivative, however, resulting in a system of equations that might be viewed as an extension to the Hall-MHD model.

TABLE I. Various limits of the ideal two-fluid plasma equations.

c	M	d_S	β	d_L	d_D	Limiting equations	Dispersion relation	Label
$<\infty$	$<\infty$	>0	>0	>0	>0	Equations (21)–(33)	Equation (41)	2FP
∞	$<\infty$	>0	>0	>0	0	Equations (64)–(71)	Equation (77)	2FMHD
$<\infty$	∞	>0	>0	>0	>0	Equations (86)–(95)	Equation (98)	1FP
$<\infty$	≥ 0	0	>0	0	0	Equations (117)–(124)	Equation (128)	QMHD
∞	∞	>0	>0	>0	0	Equations (135)–(142)	Equation (143)	Hall-MHD
∞	≥ 0	0	>0	0	0	Equations (156)–(162)	Equation (165)	Ideal MHD

Finally, the QMHD system corresponds to the $d_S \to 0$ limit. In this system, first order perturbations in the charge density and current explicitly contribute to the leading order system, unlike in the previous two. Aside from the presence of these perturbations and the displacement current, the QMHD system is remarkably similar to the single-fluid MHD model.

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APPENDIX A: DISPERSION RELATION COEFFICIENTS

As derived in Eq. (41), the non-zero polynomial coefficients for the ideal two-fluid dispersion relation are listed in Table II.

Similarly, the non-zero polynomial coefficients for the 1FP model dispersion relation given in Eq. (98) are listed in Table III.

APPENDIX B: ZERO PLASMA BETA

Here, we make a brief comment on the alternative route to achieve the small Larmor radius assumption, i.e., the $\beta \to 0$ limit. A similar perturbation procedure used throughout this paper could be attempted where series in powers of $\beta^{1/2}$ is used to expand the field variables. However, such an expansion necessarily leads to a trivial solution for all variables at zeroth order, for all sets of equations discussed in this paper, that is, a magnetized background with no motion. Physically, this agrees with the definition of plasma beta when magnetic energy dominates. However, if perturbed variables at higher order were to be extracted, equations are not closed at any truncated order.

TABLE II. Non-zero coefficients of the dispersion equation (41) for an ideal two-fluid model.

$$A_{14} = -\frac{c^2(M+1)^6(\beta c^2+2)^2}{\beta^2 d_S^6 M^3}$$

$$A_{15} = \frac{(M+1)^4(2\beta c^2+2\beta c^2M^2+M(3\beta^2c^4+4\beta c^2+4))}{\beta^2 d_S^4 M^3}$$

$$A_{16} = -\frac{(M+1)^2(3\beta c^2M+2M^2+2)}{\beta d_S^2 M^2}$$

$$A_{17} = 1$$

$$A_{23} = \frac{c^6(\frac{1}{M}+1)^6M^3\left(\beta+\frac{2}{c^2}\right)\left(\beta+2\left(\frac{1}{c^2}+1\right)\lambda^2+2\right)}{\beta^2 d_S^6}$$

$$\begin{aligned}A_{24} = &-\frac{(1+M)^4}{\beta^2 d_S^4 M^3}[2\beta^2c^6M+4\lambda^2(-\alpha+\alpha M^2+M+1)+\beta c^4(-\alpha\beta+\beta+2\lambda^2+M^2(\alpha\beta+2\lambda^2+2)+M(3\beta-2\lambda^2+6)+2)\\&+2c^2(\beta(-3\alpha\lambda^2+\alpha+3\lambda^2)+\beta M^2(3\alpha\lambda^2-\alpha+1)+2(\beta+2)M)]\end{aligned}$$

$$A_{25} = \frac{(M+1)^2(4\beta c^4M^2+c^2M(-2\alpha\beta+2\beta+2M^2(\alpha\beta+2)+3\beta M+4)+2(M+1)(\alpha M^3\lambda^2-(\alpha-1)M^2+\alpha M-(\alpha-1)\lambda^2))}{\beta d_S^2 M^3}$$

$$A_{26} = -2c^2+\tfrac{\alpha-1}{M}-\alpha M-1$$

$$A_{32} = -\frac{4c^4(M+1)^6\lambda^2((\beta+1)c^2+\lambda^2+1)}{\beta^2 d_S^6 M^3}$$

$$\begin{aligned}A_{33} = &\frac{(1+M)^4}{\beta^2 d_S^4 M^3}[-4c^2\lambda^2(M+1)(\alpha^2\beta(M+1)-\alpha(\beta+(\beta+2)M-2)-2)-4(\alpha-1)\alpha\lambda^4(M+1)^2+2\beta c^6(\lambda^2+\lambda^2M^2+M(\beta-\lambda^2+1))\\&+c^4M(-2(\alpha-1)\alpha\beta^2+4\beta+4)+c^4\beta(-\alpha^2\beta+\alpha(\beta-6\lambda^2+2)+8\lambda^2)+c^4\beta+M^2(-\alpha^2\beta+\alpha(\beta+6\lambda^2-2)+2(\lambda^2+1))]\end{aligned}$$

$$\begin{aligned}A_{34} = &-\frac{(1+M)^2}{\beta d_S^2 M^3}[\beta c^6M^2+2c^4M(-\alpha\beta+\beta+M^2(\alpha\beta+1)+2\beta M+1)-2(\alpha-1)\alpha\lambda^2(M+1)^2(M^2+1)\\&+2c^2(M+1)(-2(\alpha-1)\lambda^2+2\alpha\lambda^2M^3-(\alpha-1)M^2(\alpha\beta+2)+\alpha M(-\alpha\beta+\beta+2))]\end{aligned}$$

$$A_{35} = \frac{c^4M+2c^2(-\alpha+\alpha M^2+M+1)-(\alpha-1)\alpha(M+1)^2}{M}$$

$$A_{41} = \frac{4c^6(M+1)^6\lambda^4}{\beta^2 d_S^6 M^3}$$

$$A_{42} = -\frac{2c^2\lambda^2(M+1)^4(\beta c^4(M^2+1)-2c^2(M+1)(\alpha^2\beta(M+1)-\alpha(\beta+\beta M+M-1)-1)-4(\alpha-1)\alpha\lambda^2(M+1)^2)}{\beta^2 d_S^4 M^3}$$

$$A_{43} = \frac{c^2(M+1)^2(\beta c^4M^2+2c^2(M+1)(-(\alpha-1)\lambda^2+\alpha\lambda^2M^3-(\alpha-1)M^2(\alpha\beta+1)+\alpha M(-\alpha\beta+\beta+1))-4(\alpha-1)\alpha\lambda^2(M+1)^2(M^2+1))}{\beta d_S^2 M^3}$$

$$A_{44} = -\frac{c^2(M+1)(c^2(\alpha(M-1)+1)-2(\alpha-1)\alpha(M+1))}{M}$$

$$A_{51} = -\frac{4(\alpha-1)\alpha c^4\lambda^4(M+1)^6}{\beta^2 d_S^4 M^3}$$

$$A_{52} = \frac{2(\alpha-1)\alpha c^4\lambda^2(M+1)^4(M^2+1)}{\beta d_S^2 M^3}$$

$$A_{53} = -\frac{(\alpha-1)\alpha c^4(M+1)^2}{M}$$

TABLE III. Non-zero coefficients of the dispersion equation (98) for a 1FP model.

$D_{14} = -\frac{(\beta c^2+2)^2}{2\beta d_S^2}$
$D_{15} = 1$
$D_{23} = \frac{(\beta c^2+2)(c^2(\beta+2\lambda^2+2)+2\lambda^2)}{2\beta d_S^2}$
$D_{24} = -3\alpha\lambda^2+\alpha-\frac{1}{2}c^2(\alpha\beta+2\lambda^2+2)-\frac{2\alpha\lambda^2}{\beta c^2}-1$
$D_{25} = \frac{\alpha d_S^2\lambda^2}{c^2}$
$D_{32} = -\frac{2c^2\lambda^2((\beta+1)c^2+\lambda^2+1)}{\beta d_S^2}$
$D_{33} = \frac{2\alpha\lambda^2(-\alpha\beta+\beta+2)}{\beta}+c^4\lambda^2+c^2\left(-\frac{\alpha^2\beta}{2}+\frac{1}{2}\alpha(\beta+6\lambda^2-2)+\lambda^2+1\right) -\frac{2(\alpha-1)\alpha\lambda^4}{\beta c^2}$
$D_{34} = \alpha d_S^2\lambda^2\left(\frac{\alpha-1}{c^2}-2\right)$
$D_{41} = \frac{2c^4\lambda^4}{\beta d_S^2}$
$D_{42} = -\frac{\lambda^2(-4(\alpha-1)\alpha\lambda^2+\beta c^4+2\alpha c^2(-\alpha\beta+\beta+1))}{\beta}$
$D_{43} = \alpha d_S^2\lambda^2(-2\alpha+c^2+2)$
$D_{51} = -\frac{2(\alpha-1)\alpha c^2\lambda^4}{\beta}$
$D_{52} = (\alpha-1)\alpha c^2 d_S^2\lambda^2$

Closure may be mathematically enforced if the zeroth order magnetic field vanishes, that is, $\mathbf{B}=\mathbf{B}_1\beta^{1/2}+O(\beta)$. Such an artificial construction performs no more than rescaling the original governing equations. For example, applying the aforementioned expansion to the 2FMHD system produces the following set of equations which is identical to the rescaled 2FMHD system discussed as a remark in Sec. III A

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0, \tag{B1}$$

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\nabla p-\nabla\cdot\left(\frac{m_i^2}{e^2}\frac{\mathbf{jj}}{M\rho}\right)+\frac{\sqrt{2}}{d_S}\mathbf{j}\times\mathbf{B}_1, \tag{B2}$$

$$\frac{\partial\mathbf{j}}{\partial t}+\nabla\cdot\left(\mathbf{uj}+\mathbf{ju}-\frac{m_i(1-M)}{e\rho M}\mathbf{jj}+\frac{e(p-(M+1)p_e)}{m_i}\mathbf{I}\right) =\frac{\sqrt{2}e}{d_S m_i}\left(\frac{eM\rho}{m_i}(\mathbf{E}_1+\mathbf{u}\times\mathbf{B}_1)+(1-M)\mathbf{j}\times\mathbf{B}_1\right), \tag{B3}$$

$$\frac{\partial}{\partial t}\left(\mathcal{E}+\frac{m_i^2}{e^2}\frac{j^2}{2M\rho}\right)+\nabla\cdot((\mathcal{E}_h+p)\mathbf{u}+2\mathbf{E}_1\times\mathbf{B}_1) +\nabla\cdot\mathbf{f}_{erg}=0, \tag{B4}$$

$$\frac{\partial p_e}{\partial t}+\left(\mathbf{u}-\frac{m_i}{e\rho}\mathbf{j}\right)\cdot\nabla p_e+\gamma p_e\nabla\cdot\left(\mathbf{u}-\frac{m_i}{e\rho}\mathbf{j}\right)=0, \tag{B5}$$

$$\frac{\partial\mathbf{B}_1}{\partial t}+\nabla\times\mathbf{E}_1=0, \tag{B6}$$

$$\nabla\times\mathbf{B}_1=\frac{1}{\sqrt{2}d_S}\mathbf{j}, \tag{B7}$$

$$\nabla\cdot\mathbf{B}_1=0, \tag{B8}$$

where

$$\mathcal{E}_h=\frac{p}{\gamma-1}+\frac{1}{2}\rho u^2, \quad \mathcal{E}=\mathcal{E}_h+B_1^2, \tag{B9}$$

$$\mathbf{f}_{erg}=\frac{j^2m_i^2}{2e^2M\rho}\mathbf{u}-\frac{\gamma m_i(e(M+1)p_e\rho-ep\rho)}{(\gamma-1)e^2M\rho^2}\mathbf{j} -\frac{jm_i^2(ej(M-1)m_i-2e^2M\rho u)}{2e^4M^2\rho^2}\mathbf{j}, \tag{B10}$$

$$\mathbf{B}_1=\mathbf{B}\beta^{-1/2}, \quad \mathbf{E}_1=\mathbf{E}\beta^{-1/2}. \tag{B11}$$

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